The LIBOR Market Model and the volatility smile

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**Abstract**

The LIBOR Market Model (LLM) is a popular term structure interest rate model which lends itself to easy calibration to published market at-the-money (ATM) volatilities. Its inadequacies in explaining the interest rate volatility smile, meant that the subsequent Stochastic Alpha Beta Rho (SABR) model introduced by Hagan, Kumar, Lesniewski and Woodward in 2002 gained popularity and was adopted as a standard amongst practitioners in pricing European interest rate options. For more complex exotic instruments, modified versions of the LLM which cater for smile dynamics seem to have found some traction. This paper will present the development of the *LIBOR market model* (LLM), including a discussion on a modified LIBOR market model with *stochastic volatility* proposed by Hagan and Lesniewski.
Acknowledgements

I would like to acknowledge the supervision of Professor Barbara Swart and the support of my family who spent many hours waiting for ‘Dad’ to reappear from behind the study door. Needless to say, all errors are mine alone.
Declaration

I declare that the work I am submitting for assessment contains no section copied in whole or in part from any other source unless explicitly identified in quotation marks and with detailed, complete and accurate referencing.

(Mauro) 
(Signature)
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### Glossary

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<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ito’s Lemma</td>
<td>Ito defined the differential calculus rules that would apply to functions of infinite variation.</td>
</tr>
<tr>
<td>equivalent</td>
<td>A probability is equivalent to another if it is zero on all null sets, but non-zero otherwise.</td>
</tr>
<tr>
<td>long</td>
<td>A long position in an asset means a positive holding.</td>
</tr>
<tr>
<td>no-arbitrage</td>
<td>A condition under which there are no certain risk free profits.</td>
</tr>
<tr>
<td>predictable</td>
<td>A process $\delta(t)$ is predictable if it is adapted to the filtration $F(t)$.</td>
</tr>
<tr>
<td>risk-free</td>
<td>The rate at which governments will fund.</td>
</tr>
<tr>
<td>self-financing</td>
<td>A trading strategy is self-financing if the changes in the trading strategy are only due to changes in the tradable assets and not due to injection of additional funds.</td>
</tr>
<tr>
<td>short</td>
<td>A short position in the asset means a negative holding.</td>
</tr>
<tr>
<td>stationary</td>
<td>A process is stationary if the distribution of the increments relies on the time size of the increment.</td>
</tr>
<tr>
<td>usual conditions</td>
<td>Filtration $F(t)$ satisfies the usual conditions if $F(t)$ is right continuous and if $F(0)$ contains the null sets i.e. $f(A) = 0$.</td>
</tr>
<tr>
<td>Annuity</td>
<td>An in arrears annuity is defined as $A(i,N) = \frac{1 - \frac{1}{(1+i)^N}}{i}$ where $i$ is the interest rate for the period, and $N$ is the number of payments.</td>
</tr>
<tr>
<td>Black-76</td>
<td>Modification of the Black-Scholes equation which takes the forward price of the underlying.</td>
</tr>
</tbody>
</table>
## Acronyms

<table>
<thead>
<tr>
<th>Acronym</th>
<th>Description</th>
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</thead>
<tbody>
<tr>
<td>SVLLM</td>
<td>LIBOR Market Model with stochastic volatility.</td>
</tr>
<tr>
<td>ATM</td>
<td>at-the-money.</td>
</tr>
<tr>
<td>BBSW</td>
<td>Australian Bank Bill Swap Reference Rate.</td>
</tr>
<tr>
<td>BGM</td>
<td>Brace-Gatarek-Musiela.</td>
</tr>
<tr>
<td>BIS</td>
<td>Bank of International Settlements.</td>
</tr>
<tr>
<td>CEV</td>
<td>Constant Elasticity of Variance.</td>
</tr>
<tr>
<td>FRA</td>
<td>Forward Rate Agreement.</td>
</tr>
<tr>
<td>FRN</td>
<td>Floating Rate Note.</td>
</tr>
<tr>
<td>HJM</td>
<td>Heath-Jarrow-Morton.</td>
</tr>
<tr>
<td>LIBOR</td>
<td>London Interbank Offer Rate.</td>
</tr>
<tr>
<td>LLM</td>
<td>LIBOR Market Model.</td>
</tr>
<tr>
<td>LLM-SABR</td>
<td>Lognormal LIBOR Market Model with SABR.</td>
</tr>
<tr>
<td>LSM</td>
<td>LIBOR Swap Model.</td>
</tr>
<tr>
<td>Monte Carlo</td>
<td>Monte Carlo simulation technique.</td>
</tr>
<tr>
<td>NFLVR</td>
<td>No Free Lunch with Vanishing Risk.</td>
</tr>
<tr>
<td>OTC</td>
<td>over-the-counter.</td>
</tr>
<tr>
<td>PCA</td>
<td>normalised principal components analysis.</td>
</tr>
<tr>
<td>PDE</td>
<td>partial differential equation.</td>
</tr>
<tr>
<td>SABR</td>
<td>Stochastic Alpha Beta Rho.</td>
</tr>
<tr>
<td>SDE</td>
<td>stochastic differential equation.</td>
</tr>
<tr>
<td>USD</td>
<td>United States Dollar.</td>
</tr>
</tbody>
</table>
Symbols and Notation

\[ \mathcal{P} \] A set of equivalent measures.

\( B_t \) Bank account with parameters \( \{ t : \text{today} \} \).

\( W_{\mathbb{P}^Q} (t) \) Brownian Motion or Wiener Process with parameters \( \{ t : \text{date}; \mathbb{P}^Q : \text{Measure} \} \).

\( Z^k_t \) Deflated Price Process with parameters \( \{ t : \text{time} \} \).

\( D(t,u) \) Discount factor with parameters \( \{ t : \text{start date}; u : \text{end date} \} \).

\( E \) Expectation Operator.

\( \mathcal{F}_t \) Filtration with parameters \( \{ t : \text{time} \} \).

\( \mathbb{P}^{T+\delta} \) Forward Measure with parameters \( \{ T + \delta : \text{measure date} \} \).

\( FB_t (T,T^*) \) Forward Process with parameters \( \{ t : \text{today}; T : \text{from date}; T^* : \text{to date} \} \).

\( \mathcal{G}_t \) Gains Process with parameters \( \{ t : \text{time} \} \).

\( \sigma^2_{T,\text{cap}} (t) \) Instantaneous Black Volatility for the simple forward rate with parameters \( \{ t : \text{today}; T : \text{forward date} \} \).

\( \sigma^2_{S,T,\text{swaption}} (t) \) Instantaneous Black Volatility for the simple forward rate with parameters \( \{ t : \text{today}; S : \text{swaption start}; T : \text{swaption end} \} \).

\( \alpha_P (t,T) \) Instantaneous Bond Drift with parameters \( \{ t : \text{today}; T : \text{forward date} \} \).

\( \sigma_P (t,T) \) Instantaneous Bond Volatility with parameters \( \{ t : \text{today}; T : \text{forward date} \} \).

\( \alpha_v (\sigma,t) \) Instantaneous Drift of Volatility with parameters \( \{ t : \text{today} \} \).

\( \alpha_{FB} (t,T) \) Instantaneous Forward Process Drift with parameters \( \{ t : \text{today}; T : \text{forward date} \} \).

\( \sigma_{FB} (t,T) \) Instantaneous Forward Process Volatility with parameters \( \{ t : \text{today}; T : \text{forward date} \} \).

\( f^s_t \) Instantaneous Forward Rate with parameters \( \{ t : \text{today}; s : \text{forward date} \} \).

\( \alpha_f (t,T) \) Instantaneous Forward Rate Drift with parameters \( \{ t : \text{today}; T : \text{forward date} \} \).

\( \sigma_f (t,T) \) Instantaneous Forward Rate Volatility with parameters \( \{ t : \text{today}; T : \text{forward date} \} \).

\( r_t \) Instantaneous Short Rate with parameters \( \{ t : \text{today} \} \).

\( \alpha_f (t) \) Instantaneous Short Rate Drift with parameters \( \{ t : \text{today} \} \).

\( \sigma_f (t) \) Instantaneous Short Rate Volatility with parameters \( \{ t : \text{today} \} \).

\( J (i) \) Jump size with parameters \( \{ i : \text{index} \} \).

\( \alpha_L (t,T) \) LIBOR rate Drift with parameters \( \{ t : \text{today}; T : \text{forward date} \} \).
$\sigma_L(t,T)$: LIBOR rate Volatility with parameters $\{t : \text{today}; T : \text{forward date}\}$.

$\sigma_{\text{sabr},t}(L^t_L)$: LIBOR/SABR Volatility with parameters $\{t : \text{today}; L^t_L : \text{volatility date}\}$.

$\mathcal{C}_{\text{sabr}}(t,\sigma_{\text{sabr},t}(L^t_L),L^t_L)$: LIBOR/SABR Volatility Function with parameters $\{t : \text{today}; \sigma_{\text{sabr},t}(L^t_L) : \text{volatility}; L^t_L : \text{forward rate}\}$.

$\mathbb{M}(Z_t,\phi_t)$: Market with parameters $\{Z_t : \text{Price Process}; \phi_t : \text{Trading strategy}\}$.

$\theta(t)$: Market price of risk with parameters $\{t : \text{time}\}$.

$\lambda(t)$: Market price of volatility with parameters $\{t : \text{time}\}$.

$\lambda(t)$: Poisson Arrival Rate with parameters $\{t : \text{time}\}$.

$N_p(t)$: Poisson Process with parameters $\{t : \text{today}\}$.

$P^T_t$: Price of Zero Coupon bond with parameters $\{t : \text{today}; T : \text{redemption date}\}$.

$Z_t$: Price Process with parameters $\{t : \text{time}\}$.

$f$: Probability Symbol where usage is $f(X \in dx)$.

$\mathbb{P}$: Real World Measure.

$\mathbb{P}^Q$: Risk Neutral Measure.

$L^T_t$: Simple LIBOR forward rate at current date running from start date to start date $+ \delta$ with parameters $\{t : \text{today}; T : \text{start date}\}$.

$S_t(s,T)$: Simple Swap Rate with parameters $\{t : \text{today}; s : \text{start date}; T : \text{end date}\}$.

$\mathbb{M}^k(Z^k_t,\phi_t)$: Spot Market with parameters $\{Z^k_t : \text{Price Process}; \phi_t : \text{Trading strategy}\}$.

$\mathbb{P}^S$: Spot Measure.

$G_t$: Spot Measure Asset with parameters $\{t : \text{today}\}$.

$\eta(t)$: Tenor index of the first traded zero coupon bond occurring after time period $t$ with parameters $\{t : \text{time}\}$.

$t_T$: time with parameters $\{T : \text{tenor}\}$.

$\phi_t$: Trading strategy with parameters $\{t : \text{time}\}$.

$\mathcal{V}_t$: Variance process equal to $\sigma^2$ with parameters $\{t : \text{today}\}$.

$\mathbb{V}_t$: Wealth Process with parameters $\{t : \text{time}\}$.
Chapter 1

Introduction

1.1 Pricing Contingent Claims

Louis Bachelier’s seminal work [Bac95] on ‘The Theory of Speculation’ began a theoretical journey in applying complex mathematical models such as Brownian motion to finance. Bachelier remained unnoticed within financial academia until his contribution was brought into prominence through Paul Samuelson’s¹ work on random walks. The mathematician Leonard Savage brought Bachelier’s thesis to the attention of Samuelson, when the latter began contemplating the theory of option pricing in the 1950’s [Bac11].

The earliest models of option pricing fell into a class of absolute pricing models characterised by unobservable parameters which needed empirical estimation. Samuelson’s 1965 version required the determination of an unobservable expected rate of return on the stock, as well as an unobservable discount rate required to present value the warrant price back to the current time.

It was in 1973 that Black and Scholes [BS73] published an alternative relative pricing model characterised by the ability to perfectly hedge a self-financing portfolio. This pricing methodology derived a fundamental relationship between the derivative, its underlying and the risk-free return and relied heavily on prior work by Thorp and Kassouf [TK67]².

Later, work by Kreps and Harrison [HK79] and Harrison and Pliska [HP81] formalised a more general no-arbitrage theoretical framework in which to study the valuation of contingent claims.

1.2 Interest Rate Derivatives

In June 2012 the Bank of International Settlements (BIS) estimated the gross market value of outstanding interest rate options to be $1.848 Billion United States Dollars (USDs) (table 1.1) with the largest regions comprising of the (i) United States and (ii) the Euro Market (table 1.2). The correct pricing, valuation and hedging of interest derivatives is thus of great practical importance.

The simple stock models described by constant drift and diffusion terms of Brownian motion and later geometric Brownian motion, seemed a good historical starting point on which to price contingent claims on interest rates. The Black-76 [Bla76] modified the original Black-Scholes equation to take as input the forward stock price. Its tractability has meant that the market convention is to quote the Black-76 input implied volatility of the simple forward rates (caplets/floorlets) or swap rates (swaptions)³.

¹Samuelson has been described as the father of modern economics
²For an interesting article on the fallibility of the dynamic hedging argument the interested reader should refer to [HT09] and [Tan97].
³The Black-76 value is multiplied by different numéraires to get the dollar option price [Hau02].
Table 1.1: **BIS** - Amounts outstanding of over-the-counter (OTC) derivatives - **USD** Billions - June 2012

<table>
<thead>
<tr>
<th></th>
<th>Notional Outstanding</th>
<th>Gross Market Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total Contracts</td>
<td>638,928</td>
<td>25,392</td>
</tr>
<tr>
<td>Total Interest Rate Contracts</td>
<td>494,018</td>
<td>19,113</td>
</tr>
<tr>
<td>Forward Rate Agreements (FRA)</td>
<td>64,302</td>
<td>51</td>
</tr>
<tr>
<td>Interest Rate Swaps (IRS)</td>
<td>379,401</td>
<td>17,214</td>
</tr>
<tr>
<td>Option</td>
<td>50,314</td>
<td>1,848</td>
</tr>
</tbody>
</table>

Table 1.2: **BIS** - Interest rate derivatives by instrument, counterparty and currency - Gross market values - **USD** Millions - June 2012

<table>
<thead>
<tr>
<th></th>
<th>USD</th>
<th>Euro</th>
<th>Yen</th>
<th>GBP</th>
<th>CHF</th>
<th>CAD</th>
<th>SEK</th>
<th>Residual</th>
</tr>
</thead>
<tbody>
<tr>
<td>FRA</td>
<td>17,605</td>
<td>19,256</td>
<td>191</td>
<td>5,099</td>
<td>534</td>
<td>686</td>
<td>3,110</td>
<td>4,824</td>
</tr>
<tr>
<td>IRS</td>
<td>6,745,606</td>
<td>6,938,138</td>
<td>959,670</td>
<td>1,343,758</td>
<td>154,508</td>
<td>193,110</td>
<td>88,726</td>
<td>790,598</td>
</tr>
<tr>
<td>Option</td>
<td>623,021</td>
<td>983,330</td>
<td>95,522</td>
<td>112,837</td>
<td>5,686</td>
<td>1,338</td>
<td>2,330</td>
<td>23,832</td>
</tr>
</tbody>
</table>

Other models of interest rates were developed. Unlike observable stock prices, the first pure interest rate option models began with a theoretical construct of an unobservable instantaneous rate. The instantaneous rate models needed further parameterisation of the drift and diffusion coefficients so that the instantaneous rate reflected the complex term structure observable from spot yield curves.

The story of the **LIBOR Market Model (LLM)** began in 1992 with its predecessor the **Heath-Jarrow-Morton (HJM)** model, when Heath et al. described their term structure model based on instantaneous log-normally distributed rates. As early as 1993, Sandmann et al. proposed the replacement of the instantaneous rate with an annual effective rate, which bypassed issues of instability arising from using continuous rates. Work by Miltersen et al. followed in 1995 which suggested using London Interbank Offer Rate (LIBOR) rates. Brace, Gatarek and Musiela in 1997 described a model under the risk-neutral measure, that recovered exactly the Black-76 formula of the at-the-money (ATM) caplets. The model formulation that resulted became know as the **Brace-Gatarek-Musiela (BGM)**.

Jamshidian in 1997 followed the 1995 formulation by Musiela et al., focusing on the specification of the LIBOR rate dynamics under the spot and forward measure. Jamshidian derived the Black-76 formula for swaptions and introduced the **LIBOR Swap Model (LSM)**. The LSM and LLM were incompatible, as the composed forward swap rate did not have a lognormal distribution under the LLM. Jamshidian also suggested that the new class of term structure models be referred to as **Market Models**.

### 1.3 Outline

The main aim of this paper is to present the LIBOR Market Model incorporating a discussion on smile modelling.

The outline of the paper is as follows:

(i) In chapter the paper describes the market instruments and the analogous rate concepts.

(ii) Short rate models are introduced briefly in chapter followed by a detailed discussion on HJM in chapter.

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4. See Vasicek, Cox et al., Dothan, Longstaff et al. among others.
5. Rebonato describes the chronology of bond options and swaption/caplets/floorlets.
6. LLM nomenclature replaced some usages of BGM.
(iii) This is followed by a discussion on the LIBOR market models in chapter 5.

(iv) Chapter 6 describes practical considerations when implementing LIBOR market models such as
(i) the volatility function specification, (ii) calibration and (iii) simulation.

(v) Finally in chapter 7 the paper introduces smile modelling. We begin with a short discussion
how smile dynamics can be incorporated into a model, followed by a more detailed discussion
on stochastic volatility and the Lognormal LIBOR Market Model with SABR (LLM-SABR)
extension.

**Notation** The paper includes several formulae, including stochastic differential equations (SDEs)
and partial differential equations (PDEs). To make the formulae as unambiguous as possible a page
on Symbols and Notation is included at the beginning of the paper. Further clarification on the
SDE notion is provided by defining the uncorrelated Brownian motion in appendix D and calibrated
Brownian motion in section 6.2.1. It should be noted that the proofs in the chapter on HJM and the
Market Models are heuristic not precise.

**Appendices** The list of appendices included are

(i) risk neutral pricing (appendix A) with particular reference to model properties such as (i) exis-
tence (section A.2.2) and (ii) uniqueness (section A.2.3),

(ii) Local Volatility and Jump Models (section B),

(iii) change of measure and the relevant theorems (appendix C),

(iv) some mathematical lemmas and definitions including (i) Ito’s product and quotient rules, (ii) the
Dolean’s exponential and (iii) the Leibniz rule (appendix E), and

(v) Brownian motion (appendix D).
Chapter 2

Market Basics

2.1 Introduction

Interest rate modelling combines both traded instruments (swaps, swaptions, caplets, floorlets, zero coupon bonds) with more artificial constructs (instantaneous short rate, instantaneous forward rate, LIBOR rates). This chapter will briefly present the market instruments and discuss the interrelationship between both traded instruments and the artificial artefacts in order to lay the groundwork for later discussions on the LLM.

For a more comprehensive treatment on fixed interest products the interest reader can consult Fabossi [FM05] or Hull [Hul09], while market conventions are treated in Henrard [Hen12] and Assa [Ass12]. Pricing formulas for option pricing are comprehensively treated in Haug [Hau07].

2.2 Basics

2.2.1 Tenors and Dates

Definition 2.2.1.1 (Tenor Structure). We define a tenor structure as a set of discrete tenor dates \( t_i \subset [t_0, \ldots, t_N] \) with constant time fraction distance of \( \delta \) between successive dates.

Throughout the text we will assume the existence of a tenor structure as defined in definition 2.2.1.1. Typically the dates represented by the tenor structure will define the maturity dates of a collection of zero coupon bonds. Additionally each successive date pair will be the forward spanning dates of the forward LIBOR rates of length \( \delta \).

2.2.2 Business Day Conventions and Calendars

Generation of dates for real contracts requires both holiday calendars and business day conventions that provide rules for how dates are modified when they fall on non-business dates. Henrard [Hen12] lists the following business day conventions: (i) Following, (ii) Preceding, (iii) Modified following, (iv) Modified following bimonthly and (v) End of month.

2.2.3 Day Count

The calculation of the time fraction \( \delta \) between two dates requires the specification of a day count convention which describes the rule for calculating the time fraction in years. When modelling \( \delta \) will be assumed to be constant. Example day count conventions detailed in Henrard [Hen12] include

2.3 Linear Market Instruments

This section outlines the linear tradable instruments and relates closely to rates in section 2.5.

2.3.1 Bank/Savings Account

The bank account (savings account / money market account) is a basic instrument assumed in modelling and represents the accumulated value of one dollar deposited or loaned from a bank at time 0 held to time $t$. The Black-Scholes [BS73] framework uses the bank account as the reference risk-free asset (numéraire asset) when pricing under the risk neutral measure.

**Definition 2.3.1.1 (Bank Account).** The bank account accumulates interest at the instantaneous short rate such that a deposit of $1 at time 0 will be equal to

$$B_T = \exp \int_0^T r_u du$$

at time $T$. The change in the bank account occurs according to

$$dB_t = r_t B_t dt$$

where $B_0 = 1$.

2.3.2 Zero Coupon Bond

A zero coupon bond guarantees the holder the payment of one dollar at a maturity date $T$ for a discounted amount paid at an earlier date $t$ with no intermediate coupons. The zero coupon bond is taken as the most basic instrument in the construction of the LLM and the discounted zero coupon bond is assumed to follow a martingale process.

**Definition 2.3.2.1 (Zero Coupon Bond).** The price of a zero-coupon bond is defined as the integral of the instantaneous forward rate over the period $[t, T]$

$$P_T^t = \exp \left( - \int_t^T f_u^t du \right) \tag{2.3.1}$$

where $f^t_u$ is the instantaneous forward rate defined in definition 2.5.2.1.

2.3.3 Swaps

A swap is an agreement between two parties to exchange a stream of cash flows at particular swap frequencies over a predefined period known as the swap length.

Each leg of the swap will have a common set of features specifying how both the amount and currency of the cash flow is to be determined. Floating legs specify a floating reference interest rate which is determined at the rate-set date. Fixed legs predefine the interest rate upfront.

---

1 Swap frequencies are usually quarterly or semi-annual, but this normally depends on the swap length. In the Australian market, swaps of three years and less are quarterly, while swaps of four years and more are semi-annual.
2 The reference rate depends on both the currency and frequency of the floating leg. A three month floating leg will reference Australian Bank Bill Swap Reference Rate (BBSW) three month in Australia, three month LIBOR GBP in the UK.
This paper, when dealing with the term *swap*, will be referring to a *fixed-floating swap* with both legs denominated in the same currency such that a floating amount $N \delta L^T_t$ is exchanged for a fixed amount $N \delta K$ at each rateset date of the swap $T \subset [t_0, \ldots, t_N]$ where (i) $K$ is fixed, (ii) $L^T_t$ is the floating rate, (iii) $\delta$ is the time fraction (iv) and $N$ is the notional. The term *payer/receiver* of the swap refers to the payer/receiver of the fixed leg of the swap.

The present value of the fixed and floating legs are equal on commencement of the swap allowing the swap rate to be derived from the yield curve. The swap rate formula is the ratio of a *Floating Rate Note (FRN)* to an *Annuity* in which the floating fixings cancel with the discount factors to product two terms in the denominator. The forward swap rate $S_{t_0}(t_0,t_N)$ starting at time $t_0$ and ending at $t_N$ which values the swap at zero at $t_0$ is

$$S_{t_0}(t_0,t_N) = \frac{P^t_{t_0} - P^t_{t_N}}{\sum_{i=1}^{N-1} D(t_i,t_{i+1})}$$

where (i) $P^t_t$ is the zero coupon bond and (ii) $D(t,u)$ is the discount factor.

### 2.4 Options

This section describes the *vanilla* interest rate options such as (i) caplets/floorlets and (ii) swaptions that are traded through listed exchanges and brokers providing markets in over-the-counter (OTC) products. As volatility surfaces are readily available for these products, the success of interest rate models are to a large degree dependent on how well they calibrate to vanilla options. Traders of *exotics* will use the vanilla options to ensure that their books are *Vega* neutral.

#### 2.4.1 Caps/Floors

**Definition 2.4.1.1 (Caplet call).** A caplet call gives the holder the right but not obligation to fix a simple future forward rate at a given strike such that the pay off is described by $\max(L^t_{t_i} - K, 0)$ at time $t_i$ where (i) $L^t_{t_i}$ is a future reference rate (ii) and $K$ is the fixed strike. The market convention to price a call on a caplet is to use the Black-76 formula such that the forward price of the caplet price is

$$\frac{N \ast \tau}{1 + L^t_{t_i} \times \tau} \ast \text{Black-76}(\sigma_{t_i}, K, L^t_{t_i}, t_i)$$

where (i) $L^t_{t_i}$ is the current spot value of the reference rate, (ii) $\tau$ is the time fraction over which the reference rate applies, (iii) $N$ is the notional of the contract, (iv) $\sigma_{t_i}$ is the volatility of the reference rate, (v) $K$ is the strike and $t_i$ is the option expiry.

Caplets/floorlets are not traded individually but rather as a strip called a cap/floor. A one year ATM cap rate-setting on a three month reference rate, will consist of four options with expiry times $[3M, 6M, 9M, 12M]$ and will have a strike equal to a weighting of the forward rates.

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3 A *Cross currency swap* exchange legs of different currencies.
4 New legislation is being introduced called Dodd-Frank, which requires that swaps are collateralised. The implications are that the rate-set curve and the discounting curve will no longer be the same and the terms of the FRN denominator will no longer cancel out.
5 The Chicago Mercantile exchange trades listed interest rate option contracts.
6 See Haug [Hau07, p. 422]
2.4.2 Swaptions

A payer/receiver swaption gives the holder the right but not obligation to enter into a payer/receiver swap on option expiry. The market assumes log-normal diffusion processes for the swap rate, and quotes the Black-76 implied volatilities (see [Hau07] for a detailed descriptions of their respective formulas). The ATM strikes of short dated swaptions are the most liquid, with the skews quoted at various option delta’s around the ATM. Swaptions may be cash settled for some currencies such as EUR or GBP.

2.5 Rates

This section introduces the reader to conceptual interest rate quantities such as instantaneous rates as well as market quoted rates such as LIBOR.

2.5.1 Compounding, Frequency and Quotation method

Quotation conventions are needed to augment a rate percentage in order to provide a meaningful interpretation when the rate is used to calculate a dollar interest. If we are given a priori a fixed rate \( r \), a time period \([t_1,t_2] \) and time fraction \( \tau \) calculated using the day count convention in section 2.2.3 then the value of one dollar over the period using different compounding techniques would be:

(i) \( \exp (r\tau) \) under continuous compounding,
(ii) \((1 + r)^\tau \) with annual compounding,
(iii) \((1 + \frac{r}{2})^{\tau\cdot2} \) using semi-annual compounding,
(iv) \((1 + \frac{r}{n})^{\tau\cdot n} \) using \( n \)-frequency compounding,
(v) and \( 1 + \tau r \) under simple compounding.

2.5.2 Instantaneous Rates

The conceptual interest rates introduced in this section are a precursor to understanding short rate models and HJM.

**Definition 2.5.2.1 (Instantaneous Short Rate).** The instantaneous short rate \( r_t \) is the continuously compounded rate at which cash can be loaned or borrowed for an infinitesimal period at time \( t \).

**Definition 2.5.2.2 (Instantaneous Forward Rate).** The instantaneous forward rate \( f_s^t \) is the forward continuously compounded rate contracted at time \( t \leq s \) at which cash can be instantaneously loaned or borrowed for some future time \( s \geq t \).

The instantaneous short and forward rates relate to each other via the equation \( r_t = f_s^t \).

We can now back out the instantaneous forward rate by taking the partial derivative of the log of the zero coupon bond with respect to the maturity date \( T \) to get:

\[
f_T^t = -\frac{\partial \log (P_T)}{\partial T}.
\]

\( f_T^t \) \( \quad \) (2.5.1)

\( ^7 \) A 10x15 payer swaption, is the right to a 15 year payer swap in ten years time.
2.5.3 LIBOR Forward Rate

**Definition 2.5.3.1** (forward LIBOR). The simple LIBOR forward rate between \( t_i \) and \( t_{i+1} \) of length \( \delta \) is defined in terms of the zero coupon bonds as

\[
L_{t_i}^{t_{i+1}} = \frac{P_t^{t_i} - P_t^{t_{i+1}}}{\delta P_t^{t_{i+1}}}. \quad (2.5.2)
\]

Another representation is

\[
1 + \delta L_{t_i}^{t_{i+1}} = \frac{P_t^{t_i}}{\delta P_t^{t_{i+1}}}. \quad (2.5.3)
\]

The simple LIBOR forward rate relates to the instantaneous forward rate via

\[
1 + \delta L_{t_i}^{t_{i+1}} = \exp \left( \int_{t_i}^{t_{i+1}} f_s \, ds \right). \quad (2.5.4)
\]

**Definition 2.5.3.2** (Bond Index). Define the function \( \eta(t) \) as the index of the first tenor \( t_i \) in the tenor structure (definition 2.2.1.1) after the time \( t \).

Bonds can be constructed from the simple LIBOR forward rates through the following relationship

\[
P_t^{t_N} = P_t^{t_{\eta(t)-1}} \times \prod_{i=\eta(t)}^{N-1} \frac{1}{(1 + L_{t_i}^{t_{i+1}})} \quad (2.5.5)
\]

where \( \eta(t) \) is defined as in definition 2.5.3.2.
Chapter 3

Short Rate Models

3.1 Introduction

This section will describe the short rate models which were the first attempts to solve the problem of
modelling interest rates in a no-arbitrage setting. The first short rate models followed on from Merton
\[\text{Mer73}\] who in 1973 introduced a stochastic discount factor into the Black-Scholes analysis. In this
work Merton postulated that the short rate followed a stochastic process of the form
\[\begin{align*}
dr_t &= a dt + \sigma dW^S(t)
\end{align*}\]
where (i) \(a\) and \(\sigma\) were constants (ii) and \(W^S(t)\) was a Brownian motion under the spot measure.

3.2 Dynamics

Short rate models postulated the form of the short rate dynamics under the very general form
\[\begin{align*}
dr_t &= \alpha_r(t) dt + \sigma_r(t) dW^{PQ}(t) \\
\end{align*}\]
(3.2.1)
where (i) \(\alpha_r(t)\) was the drift term, (ii) \(\sigma_r(t)\) was the diffusion term (iii) and \(W^{PQ}(t)\) was the Brownian
motion under the risk neutral measure \(P^Q\). The exact form of the coefficients \((\alpha_r(t)\) and \(\sigma_r(t)\)) differed
according to the specification of the model.

Vasicek \[\text{Vas77}\] developed the first short rate model in 1977 where he postulated the use of a mean
reverting Ornstein-Uhlenbeck \[\text{UO30}\] process of the form
\[\begin{align*}
dr_t &= a (b - r_t) dt + \sigma_r(t) dW^{PQ}(t) \\
\end{align*}\]
(3.2.2)
where (i) \(b\) was the long term mean of the instantaneous short rate, (ii) \(a\) is the rate of reversion
(iii) and \(\sigma_r(t) = \sigma\).

The subsequent work by Cox, Ingersoll and Ross \[\text{CIJR85}\] introduced the CIR model which solved
the problem of negative rates in the Vasicek model by introducing a square-root term such that the
equation became
\[\begin{align*}
dr_t &= a (b - r_t) dt + \sigma_r(t) d\sqrt{r_t} W^{PQ}(t) \\
\end{align*}\]
(3.2.3)
Some short rate models were more analytical tractability than others and admitted an affine form for
the bond price equation. If the zero coupon bond had an affine pricing formula then the bond was
linearly related to the short rate \(r_t\) by the formula
\[\begin{align*}
P^T_t = \exp^{A(t,T) - B(t,T)r_t}.
\end{align*}\]
Bjork \[\text{Bjo04}\] derives the general restrictions on SDE coefficients to produce affine models. Duffie
and Kan \[\text{DK96}\] investigated the restrictions on the coefficients in relation to multi-factor short rate
models.
3.3 Classification of Short Rate Models

It becomes apparent that the driving force behind the chronological evolution of models (from Merton [Mer73] through to present day) has been the continuous need to overcome shortcomings inherent in the models. Some of the most important considerations that drove the development and evolution of short rate models were:

(i) the avoidance of negative rates,
(ii) the incorporation of equilibrium concepts such as long term rates
(iii) and the behaviour of the model in relation to the empirical yield curve which manifested itself through incorrect calibration or inconsistent shift dynamics.

This in turn provides us with a suitable taxonomy for describing our rates models based on (i) factors, (ii) time parameterisation (iii) and equilibrium versus no-arbitrage type models.

Factors. Musiela and Rutkowski [MR05] suggest that short rate models be classified by the number of driving sources of randomness (factors) and the implied number of state variables. The first classification refers to the dimensionality of the Brownian motion classifying short rate models into either Single Factor (parallel shocks) or Multi-Factor models. The second classification refers to the number of state variables required to model the embedded stochastic processes, where Markovian processes would require the smallest set of state variables.

Time parameterisation. Short rate models can be discriminated by (i) constant (time-homogeneous) (ii) and time varying (time-inhomogeneous) parameterisation. The first models such as Vasicek [Vas77], Dothan [Dot78] and Cox, Ingersoll and Ross [CIJR85] were time homogeneous and used constant coefficients. The need to improve the fit against the yield curve gave rise later to the time-inhomogeneous models of Black-Derman-Toy [BDT90], Ho-Lee [HL86] and Hull-White (which incorporated versions of both the Vasicek and CIR).

Equilibrium versus No-arbitrage Models. Equilibrium models incorporated behavioural assumptions about economic variables such as the long term average short rate. In section 3.2 we briefly mention the two equilibrium models of Vasicek and CIR which assumed the short rate would drift towards a long term average rate. Hull [Hul09, p. 678] discusses how equilibrium models only approximate the yield curve, as opposed to no-arbitrage models with time-inhomogeneous parameters which will fit to a given yield curve taken as input.

3.4 No-arbitrage Short Rate Models

It should be noted that unlike the stock, the instantaneous short rate did not conceptually equate to a price of a tradable asset. The original Black-Scholes model included both a Bank Account and Stock with which to replicate the derivative, however the interest rate model setting only included the Bank Account and an exogenously specified short rate. Bonds were derivatives of the short rate and in fact not first class citizens. Thus the specification of the short rate dynamics and the requirement that the zero coupon bond was arbitrage free was not sufficient to uniquely price a derivative.

Applying no-arbitrage analysis to an interest rate setting as detailed in Bjork [Bjo04, p. 368] we can prove that there exists a market price of risk

\[ \theta(t) = \frac{\alpha P(t,T) - r_t}{\sigma P(t,T)} \]
which holds for all bonds of length $T \subset [t_0, \ldots, t_N]$. Here (i) $r_t$ is the short rate, (ii) $\alpha_P(t,T)$ is the return on the bond and (iii) $\sigma_P(t,T)$ is the volatility of the bond.

Furthermore we can derive the interest rate no-arbitrage PDE which must hold for all bonds and we state this in proposition 3.4.0.1 [Bjo04] without proof (See Bjork [Bjo04] for details.).

**Proposition 3.4.0.1 (Interest Rate No-Arbitrage PDE / Term Structure Equation).** In an arbitrage free bond market, the bond $P^T_t$ of length $T$ will satisfy the following no-arbitrage condition:

$$\frac{\partial P^T_t}{\partial t} + \left( \alpha_P(t,T) - \theta(t) \sigma_P(t,T) \right) \frac{\partial P^T_t}{\partial r_t} + \frac{1}{2} \sigma_P(t,T)^2 \frac{\partial^2 P^T_t}{\partial r_t \partial T} - r_t P^T_t = 0 \quad (3.4.1)$$

where: (i) $\theta(t)$ is the market price of risk, (ii) $r_t$ is the instantaneous short rate, (iii) $\alpha_P(t,T)$ and $\sigma_P(t,T)$, (iv) and $P^T_T = 1$.

The market price of risk term $\theta(t)$ appears in the term structure equation (3.4.1) and has important implications for our analysis of interest rates. It means that only specifying a short rate SDE will leave an interest rate model under constrained and incomplete. Additionally to the SDE, $\theta(t)$ will need to be exogenously specified or inferred from the market price of bonds. See Bjork [Bjo04, p. 371] for the details of the argument.

### 3.5 Inversion of the Yield Curve

As was stated earlier in section 3.4, specification of an SDE is not enough to define an interest rate model. At least one bond must be taken as exogenous to infer the $\theta(t)$ and create a complete market. Empirically this requires the calibration of the interest rate model to market data by iteratively changing the SDE parameters ($\alpha_r(t)$ and $\sigma_r(t)$) until the implied bond prices $\{P^T_t : 0 \leq t \leq T\}$ match that of the empirical market bond prices $\{P^{T*}_t : 0 \leq t \leq T\}$.

Calibration to the spot yield curve in short rate models can be cumbersome and difficult. The exact fit between empirical and calculated values is constrained by the form of the parameterisation of the coefficients. This meant that sometimes models would not calibrate correctly to the yield curves implied by bond prices $\{P^{T*}_t : 0 \leq t \leq T\}$. The inversion of the yield curve is described in more detail in [Bjo04, p. 376] and [BM07, p. 54].
Chapter 4

Heath-Jarrow-Morton

4.1 Introduction

The drawbacks according to Bjork [Bjo04] of the short rate models were:

(i) their inability to describe realistic volatility structures for the forward rates,

(ii) their difficulty in calibrating to the yield curve under more complex parameterisations

(iii) and their simplistic assumption of a single causal explanatory economic variable for yield curve

 behaviour (at least for single factor models).

To combat this, Heath et al. [HJM92] developed a framework in 1992 that described the evolution

\( \{ f(t,T), 0 \leq t \leq T \leq T^* \} \)

up to a fixed maturity date \( T^* \) in terms of the

instantaneous forward rates (definition 2.5.2.2). Heath et al.’s continuous-time term structure model

was built upon the previous work by Ho and Lee [HL86] who had constructed a discrete-time binomial

tree term structure model.

The [HJM] model diverged from the other continuous-time models of the time that described the yield curve in terms of a single instantaneous short rate (definition 2.5.2.1). A key result established by

Heath et al. was that the drift term was no longer independent of the diffusion term as was the case with short rate models.

Conveniently under certain assumptions on the form of the volatility function \( \sigma_f(t,T) \), the HJM

framework coincided with existing short rate models such as (i) Ho-Lee [HL86], (ii) Vasicek [Vas77]

\footnote{For constant \( \sigma_f(t,T) \) and a single driving Brownian motion HJM corresponded to the Ho-Lee Model with calibrated drift.} and (iii) Hull and White [HW90] models. Restrictions on the form of the volatility function to allow

Markovian like calculations are discussed in Ritchken et al. [RS95].

An important property of the [HJM] model was its simplicity to calibrate. The model calibration was

assured by: (i) defining the initial forward curve \( \{ f(t,u), 0 \leq t \leq u \leq T^* \} \) to be consistent with

the traded bonds and (ii) ensuring that the volatility function \( \sigma_f(t,T) \) was consistent with traded

derivative securities.

4.2 No-arbitrage Formulation of the HJM

The HJM played a pivotal role in the construction and development of the Market Models discussed

in chapter 5. The derivation, although technical, mainly involves stating the terminology needed in

\footnote{For exponential \( \sigma_f(t,T) \) HJM corresponded to the Vasicek model with time varying drift.}
construction of the domain. Glasserman [Gla03] or Musiela et al. [MR05, p. 386] can be consulted for more details.

We begin with stating some assumptions and lemmas and follow this with an informal proof (proof 4.2).

**Assumption 4.2.0.2 (HJM forward rate dynamics).** Heath et al. [HJM92] began with the assumption that instantaneous forward rate dynamics were modelled exogenously over a period \( t \in [0,T^*] \) by

\[
df_t = \alpha_f(t,T)\,dt + \sigma_f(t,T)^T\,dW^P(t)
\]

under the real world measure \( P \) where

(i) both the (i) drift term \( \alpha_f(t,T) \) \( \{ (t,T) : 0 \leq t \leq T \leq T^* \} \times \{ \omega \in \Omega \} \to \mathbb{R} \) and (ii) volatility term \( \sigma_f(t,T) \) \( \{ (t,T) : 0 \leq t \leq T \leq T^* \} \times \{ \omega \in \Omega \} \to \mathbb{R}^d \) were adapted to the filtration \( \mathcal{F}_t \) (definition A.2.1.1) generated by the Brownian motion \( dW^P(t) \) (definition D.0.5.1)

(ii) and the volatility \( \sigma_f(t,T) \) was exogenously specified with initial condition defined by (i) the initial forward curve \( \{ f(t,u) : 0 \leq t \leq u \leq T^* \} \).

**Lemma 4.2.0.3 (HJM bond dynamics).** Apply Ito’s Lemma to the zero coupon bond formula (2.3.1) to get the dynamics of the zero coupon bond in terms of the forward rate such that

\[
dP_t^T = \alpha_P(t,T)\,P_t^T\,dt + \sigma_P(t,T)\,P_t^T\,dW^P(t)
\]

where the coefficients are given by

\[
\alpha_P(t,T) = f_t^T - \int_t^T \alpha_f(t,u)\,du + \frac{1}{2} \left| \int_t^T \sigma_f(t,u)\,du \right|^2
\]

and

\[
\sigma_P(t,T) = -\int_t^T \sigma_f(t,u)\,du
\]

**Definition 4.2.0.3 (Forward Process).** Define the forward process as the ratio of two zero coupon bonds such that

\[
FB_t(T,T^*) = \frac{P_t^T}{P_t^{T^*}} \text{ for all } t \in [0,T \wedge T^*].
\]

The dynamics of the forward process is given by the application of lemma E.0.6.2 to (4.2.5) to get

\[
dFB_t(T,T^*) = FB_t(T,T^*) \left( \alpha_{FB}(t,T)\,dt + \sigma_{FB}(t,T)\,W^P(t) \right),
\]

where (i) the coefficients are given by

\[
\alpha_{FB}(t,T) = \alpha_P(t,T) - \alpha_P(t,T^*) - \sigma_P(t,T) (\sigma_P(t,T) - \sigma_P(t,T^*))
\]

and

\[
\sigma_{FB}(t,T) = \sigma_P(t,T) - \sigma_P(t,T^*)
\]

and (ii) \( W^P(t) \) is the Brownian motion under a real world measure \( P \).

Heath et al. formulation under the Forward Martingale Measure. (i) Begin with assumption 4.2.0.2 on the form of the forward rate dynamics. (ii) Assume the existence of a finite set of bonds over a given tenor structure (definition 2.2.1.1). (iii) Using the relationship between the zero coupon bond and instantaneous forward rates (2.3.1), derive the bond dynamics in terms of the forward rate dynamics as stated in lemma 4.2.0.3. (iv) Introduce an auxiliary process \( FB_t(T,T^*) \) (definition 4.2.0.3) defined as the ratio of two zero bonds of length \( T \) and \( T^* \). The forward process will be a martingale under a risk neutral measure and will constrain the relationship between the coefficients. (v) Use Ito’s
quotient rule (lemma E.0.6.2) to derive the dynamics of the forward process in terms of the bond dynamics (lemma 4.2.0.3). (vi) Now apply Girsanov’s Theorem (theorem C.0.4.2) to the forward process (4.2.3). Assume the existence of a function \( \theta(t) \) (the market price of risk) such that all forward processes follow martingales. This leaves us with the drift adjusted version of (4.2.7) which is omitted in this proof. (vii) Now taking the \( dt \) term of the martingale adjusted forward process and equating this to zero we are left with the equality

\[
\alpha_P(t,T) - \alpha_P(t,T^*) - (\sigma_P(t,T^*) - \theta(t)) (\sigma_P(t,T) - \sigma_P(t,T^*)) = 0. \tag{4.2.9}
\]

(viii) Differentiating (4.2.9) with respect to \( T \) we are left with the final result stated in proposition 4.2.0.4.

**Proposition 4.2.0.4** (HJM no-arbitrage condition). There exists a no-arbitrage relationship between the drift and diffusion terms of the forward rate such that:

\[
\alpha_f(t,T) = -\sigma_f(t,T) \left( \theta(t) + \int_T^{T^*} \sigma_f(t,u) \, du \right) \tag{4.2.10}
\]

where (i) \( \theta(t) \) is the market price of risk used in the Girsanov theorem and (ii) \( 0 \leq t \leq T \leq T^* \).

### 4.3 Other Topics

#### 4.3.1 Gaussian Derivation

Under constant parameterisation of the drift and diffusion terms, we can reduce the HJM model to a Gaussian model. Under this simplified analysis, the market price of risk is no longer relevant and drops out of the equations. Musiela et al. [MR05, p. 392] can be consulted for a more rigorous account.

#### 4.3.2 Musiela Parameterisation

A slightly different parameterisation called the ‘Musiela Parameterisation’ replaces the absolute time to maturity \( T \) by time remaining \( T - t \). It is useful to analyse the geometric properties of the HJM model. See Musiela et al. [MR05, p. 429] for details.
Chapter 5

LIBOR Market Models

5.1 Introduction

The ‘market models’ were developed as a response to several shortcomings of the HJM model including:

(i) the positive possibility of an explosion of the forward rate and
(ii) that models needed to align better with how prices were quoted in the market.

Several authors were involved in the evolution of the market models and we detail some of these derivations below in section 5.2.

5.2 Derivation

5.2.1 Miltersen, Sandmann and Sondermann

For particular forms of the HJM instantaneous volatility function, as in when it is proportional to the current value of the forward rate as in

\[ \alpha_f (f(t,T),t,T) = \sigma_f (t,T) f(t,T), \]

there is a positive probability that the forward rates will explode. Work by Miltersen et al. [MSS97] in 1997 addressed this issue by introducing a model for the simple rate.

Assumption 5.2.1.1 (Miltersen et al. LIBOR assumption). Miltersen et al. postulated that the simple rate process followed the dynamics

\[ dL^T_t = \alpha_L (t,T) dt + L^T_t \sigma_L (t,T) dW^P_{\mathbb{Q}} (P) \]

with (i) a deterministic volatility function \( \sigma_L (t,T) \).

Using assumption 5.2.1.1 Miltersen et al. derived a PDE for the bond option price.

5.2.2 Brace, Gatarek and Musiela

Miltersen et al. did not definitively establish the existence of the family of LIBOR rates. This was not done until Brace et al. [BM+97] described a model constructed over the tenor structure \( \{ T : 0 \leq T \leq T^* \} \) for a positive delta \( \delta > 0 \).

Using the HJM as a base, the authors derived the dynamics of the simple LIBOR rate \( L^T_t \) under the risk neutral probability measure \( \mathbb{P}^Q \). We lay out assumptions and definitions first, followed by proof 5.2.2.
Assumption 5.2.2.1 (BGM Volatility Function). Assume the exogenous specification of an LIBOR rate volatility function $\sigma_L(t,T)$.

Assumption 5.2.2.2 (Forward LIBOR Bond Condition). Define a family of strictly decreasing bonds $\{T : 0 \leq T \leq T^*\}$ with associated forward LIBOR rate

$$1 + \delta L^T_t = \frac{P^T_t}{P^{t+1}_t} \quad (5.2.2)$$

subject to (i) the initial forward curve $1 + \delta L^T_0 = \frac{P^T_0}{P^{0+1}_0}$ and (ii) $L^T_0 > 0$.

Assumption 5.2.2.3 (BGM LIBOR dynamics). Bruce et al. postulated the form of the LIBOR dynamics under the real world measure $\mathbb{P}$ as:

$$dL^T_t = \alpha_L(t,T) \, dt + L^T_t \sigma_L(t,T) \, dW^P_t \quad (5.2.3)$$

where (i) $\sigma_L(t,T)$ is exogenously given by assumption 5.2.2.1 and (ii) $\alpha_L(t,T)$ is left unspecified.

Definition 5.2.2.1 (LIBOR forward rate relationship). The LIBOR rate and forward rate are related via the following equation

$$1 + \delta L^T_t = \exp \left( \int_{t}^{t+1} f^*_s \, ds \right). \quad (5.2.4)$$

BGM derivation. (i) Start with assumption 5.2.2.2 around the LIBOR rates and assumption 5.2.2.1 around the volatility function. (ii) Now postulate the LIBOR dynamics given in definition 5.2.2.3 under a real world measure. (iii) Apply Ito’s lemma and the Leibniz integral rule in definition E.0.6.4 to both sides of (5.2.4) using the HJM specification for instantaneous forward rate dynamics $df^*_t$ given in (4.2.1). The results are

$$\frac{\delta d \left( L^T_t \right)}{1 + \delta L^T_t} = \frac{(1 + \delta L^T_T)}{1 + \delta L^T_0} \left( \int_T^{T+ \delta} \sigma_f(t,u) \, du \right) dW^P_t + (\ldots) \, dt$$

(iv) Comparing the $W^P(t)$ terms, we have

$$L^T_t \sigma_L(t,T) = (1 + \delta L^T_t) \left( \int_T^{T+ \delta} \sigma_f(t,u) \, du \right)$$

and what follows easily

$$\frac{L^T_t \sigma_L(t,T)}{1 + \delta L^T_t} = \left( \int_0^{T+ \delta} \sigma_f(t,u) \, du - \int_0^T \sigma_f(t,u) \, du \right).$$

(v) By taking $\sigma_f(t,T) = 0$ for $T < t + \delta$, we can use forward induction to calculate $\sigma_f(t,t_j)$ for the maturity $t_j > T$. Here we look at a derivation with $\sigma_f(t,T+ \delta) = \int_0^{T+ \delta} \sigma_f(t,u) \, du$

$$\sigma_f(t,T+ \delta) = (\sigma_f(t,T+ \delta) - \sigma_f(t,0)) + \ldots + (\sigma_f(0,T+ \delta) - \sigma_f(0)) \left( \frac{L^T_0 \sigma_L(0,T)}{1 + \delta L^T_0} - 0 \right)$$

(vi) Using the HJM relationship to $\alpha_L(t,T)$, we can calculate the drift term and finally specify the full dynamics. □
In the construction by Brace et al., the function $\sigma_L(t,T)$ is constructed as a piecewise continuous however non-differentiable function which violates the condition of differentiability of $\sigma_L(t,T)$ and thus of $\sigma_f(t,T)$ imposed by HJM. The final result is given in proposition 5.2.2.4.

**Proposition 5.2.2.4 (BGM LIBOR dynamics).** The LIBOR rate $L^j_t$ satisfies the dynamics

$$dL^j_t = L^j_t \sigma_L(t,t_j) \left( \sum_{k=0}^{\eta(t_j-\delta)} \frac{\delta \sigma_L(t,t_k)}{1 + \delta L^j_t} + dW^Q(t) \right),$$

where (i) $W^Q(t)$ is the Brownian motion under the risk neutral measure with the bank account $B_t$ as the numéraire and (ii) $\sigma_L(t,t_j)$ exogenously specified.

### 5.2.3 Jamshidian

Jamshidian [Jam97] derived the LSM which priced interest rate swaptions in line with the market assumption of a log-normal distribution for the swap rate. To do this, he introduced an accrual factor numéraire which appears as the numerator in (2.3.2). He then postulated that the bond would follow a martingale process under this numéraire.

When describing the LSM Jamshidian [Jam97] proposes three classes of swaptions to facilitate the construction of the LSM: (i) co-initial have the same start date, with varying swap length, (ii) co-terminal have the same end date, with varying swap length and (iii) co-sliding have the same swap length but varying start/end dates. The LSM was not consistent with the LLM model as both (i) forward LIBOR and (ii) swap rates cannot both be log-normal.

Jamshidian was also responsible for modelling the forward LIBOR rates under the spot martingale measure in definition 5.2.3.3. Below we introduce definitions and concepts followed by proof 5.2.3.

**Definition 5.2.3.1 (Rolling bond numéraire).** Define a rolling bond process such that

$$G_t = P^{t_{\eta(t)}}_t \prod_{j<\eta(t)} P^j_t. \quad \text{(5.2.5)}$$

where (i) $\eta(t)$ is defined as in definition 2.5.3.2 and (ii) $P^{t_{\eta(t)}}_t$ is the first bond after the current time $t$.

**Definition 5.2.3.2 (Spot Discounted Bond).** Define a process called the spot discounted bond which is the zero coupon bond divided by the rolling bond definition 5.2.3.1

$$P^T_t G_t.$$

**Definition 5.2.3.3 (Spot Martingale Measure).** The spot martingale measure is the probability $\mathbb{P}^S$ under which the process $P^T_t G_t$ is a local martingale. The numéraire $G_t$ can be interpreted as investing in the next maturing bond $t_{\eta(t)}$ and on maturity rolling this into the bond with maturity $t_{\eta(t)+1}$.

**Assumption 5.2.3.1 (Bond dynamics).** If we assume the following bond dynamics

$$dP^T_t = \alpha_P(t,T) P^T_t dt + \sigma_P(t,T) P^T_t dW^Q(t) \quad \text{(5.2.6)}$$

exist under the measure $\mathbb{P}$.

**Jamshidian LMM derivation.** (i) The spot discounted bond follows a local martingale under the spot measure in definition 5.2.3.3 (ii) Applying lemma E.0.6.2 to the above process using what we have assumed about the bond dynamics in assumption 5.2.3.1 and the adapted process $\theta(t)$ (market price of risk), we get the martingale condition

$$\alpha_P(t,t_j) - \alpha_P(t,t_{\eta(t)}) = (\sigma_P(t,t_{\eta(t)}) - \theta(t)) \left( \alpha_P(t,t_j) - \alpha_P(t,t_{\eta(t)}) \right).$$
(iii) Applying Ito’s lemma to definition 5.2.3.2 and using some further manipulation [MR05, p. 457] we derive the dynamics of the forward LIBOR process in proposition 5.2.3.2. The LIBOR drifts under the k-forward measure is stated later under the LLM-SABR in (7.4.4).

**Proposition 5.2.3.2 (Jamshidian LIBOR dynamics).** The LIBOR rate $L_{t}^{t_{j}}$ satisfies the dynamics

$$dL_{t}^{t_{j}} = L_{t}^{t_{j}} \sigma_{L}(t,t_{j}) \left( \sum_{k=\eta(t_{j})}^{j} \frac{\delta_{L}(t,t_{k})}{1 + \delta L_{t}^{t_{j}}} + dW_{t}^{L}(t) \right)$$

where (i) $L_{t}^{t_{j}}$ is the Brownian motion under the spot measure with the rolling bond definition 5.2.3.1 as the numéraire.
Chapter 6

LIBOR Market Models in Practice

6.1 Introduction

According to Rebonato [RW09] the extended LIBOR market models are the market norm for pricing exotics while the Stochastic Alpha Beta Rho (SABR) model is used for pricing European interest rate options. However models generally only provide a loose framework which act more as a guide than an exact specification. The implementation of an interest rate model therefore requires a series of unambiguous choices which may or may not be completely supported by theory.

Jamshidian [Jam97] suggests that theory should be used as a guide, with sensible improvisations. Taleb [Tal97] provides an insightful guide into how traders’ treat option models.

The market model framework requires additional qualification before it can be applied. This section describes these qualifications namely: (i) the specification of the stochastic component in section 6.2, (ii) the calibration to the market prices in section 6.3 and (iii) simulation under the martingale measures in section 6.4.

6.2 Stochastic Component

In practice the diffusion term of the LIBOR market model SDE (\( \sigma W \)) in (5.2.3) can be interpreted in different ways. In section 6.2.1 we outline choices for \( W \), while section 6.2.2 describes common choices for the volatility function.

6.2.1 Correlated Brownian motion

Up until now, our discussion around Brownian motion has involved the d-dimensional process we defined in appendix D.0.5. However this formulation is not convenient in practice when calibrating from (i) implied volatilities and (ii) historical/market correlations. In definition 6.2.1.1 we define the correlated Brownian motion which is used in practice when calibrating to the caplet and swaption volatility quotes. The relationship to its d-dimensional counterpart is given in appendix D.0.6

**Definition 6.2.1.1** (Correlated Brownian Motion). Let \( X_i \) satisfy equation (D.0.1), then we define the correlated Brownian motion \( \hat{W}_i \) as a scalar variable such that \( X_i \) now satisfies

\[
\text{d}X_i(t) = \ldots + \hat{\sigma}_i(t) \text{d}\hat{W}_i(t)
\]

where

(i) the Brownian motions are correlated by \( \rho_{i,j} \) such that \( \mathbb{E}[\text{d}\hat{W}_i, \text{d}\hat{W}_j](t) = \rho_{i,j} \text{d}t \) for all \( i \neq j \),

(ii) the two functional forms of \( \sigma \) are related via \( \hat{\sigma}_i(t) = [\sigma_{i,1}, \ldots, \sigma_{i,d}]|\sigma_{i,1}, \ldots, \sigma_{i,d}|^{-1} \)
\( \text{(iii) and the new Brownian motion relates to the old one via } \dot{W}_i(t) = \int_0^t \dot{\sigma}_i(u) \, dW_i(t) \). 

### 6.2.2 Volatility Function

#### Introduction

Market models are not prescriptive of the form of the volatility function \( \sigma \), however model implementers will need to make assumptions around the behaviour of the volatility surface. Ignoring the strike dimension and the related smile issue in market models, the volatility function needs to reflect the market volatility term structure subject to the ‘norm’ condition in definition 6.2.2.1.

**Definition 6.2.2.1 (Norm Condition).** The quoted Black market volatility \( \sigma_{\text{black},t,T} \) for a caplet ending at time \( T \) relates to the volatility function via the ‘norm’ condition:

\[
\sigma_{\text{black},t,T}^2 \times T = \int_0^T \sigma(u, \ldots)^2 \, du. \tag{6.2.1}
\]

The form of the SDE coefficients (time homogeneous, constant) will impact the distributional properties (Markovian, Gaussian etc...) which in turn impact calculation and calibration choices such as (i) simulation, (ii) PDE approaches, (iii) closed form solutions or (iv) approximation techniques. For analysis it is often convenient to restrict the instantaneous volatility function \( \sigma(\ldots) \) to be piecewise flat forward so that solutions exhibit Gaussian or Markovian properties.

A taxonomy for the volatility function centres around the differentiation based on (i) parameter choices (time to maturity, absolute time, spot) and (ii) separation of the volatility function into a correlation \( \rho(\ldots) \) and volatility function \( \hat{\sigma}(\ldots) \). We discuss only the former below.

#### Specification

We separate the volatility function into time segments such that

\[
\sigma_{\text{black},t,T}^2 \times T = \int_{t_0}^{t_1} \sigma_{1,t_0}(u)^2 \, du + \ldots + \int_{T-\delta}^{T} \sigma_{1,T-\delta}(u)^2 \, du \tag{6.2.2}
\]

where (i) \( t_0 < t_1 < \ldots < T \) and (ii) \( \sigma_{a,b}(u) \) is the instantaneous volatility function for the forward rate fixing at time \( t_a \) in the time segment \( u \subset [b,b+\delta] \). In table 6.1 we construct a tabular representation of (6.2.2) where we represent a different LIBOR rate in each row broken down into its constituent instantaneous volatility functions over the period until it expires. The instantaneous volatility function \( \sigma_{2,t_2}(u) \) is assumed to potentially differ over distinct time intervals \([t_0,t_1], \ldots, [T-\delta,t_T]\) in which the LIBOR rate is ‘alive’.

<table>
<thead>
<tr>
<th>Forward Rate</th>
<th>Period</th>
<th>( [t_0,t_1] )</th>
<th>( [t_1,t_2] )</th>
<th>( [t_2,t_3] )</th>
<th>( [t_3,t_4] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( [t_0,t_1] )</td>
<td>NA</td>
<td>( \sigma_{1,t_0}(u) )</td>
<td>NA</td>
<td>NA</td>
<td></td>
</tr>
<tr>
<td>( [t_1,t_2] )</td>
<td>( \sigma_{1,t_1}(u) )</td>
<td>( \sigma_{2,t_1}(u) )</td>
<td>NA</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( [t_2,t_3] )</td>
<td>( \sigma_{1,t_2}(u) )</td>
<td>( \sigma_{2,t_2}(u) )</td>
<td>( \sigma_{3,t_2}(u) )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 6.1: Forward Rate Separable Volatility Functions

\(^1\)NA means that the period does not apply to the respective LIBOR rate.
Our summary of the volatility specification in Table 6.1 is very verbose. In order to reduce the number of component choices of the volatility function we can make some simplifying assumptions around the relationships between the volatility segments. Brigo et al. [BM07, p. 210], Rebonato [Reb05, p. 667] and Musiela et al. [MR05, p. 413] suggest simplifications where:

(i) the volatility function depends on time-remaining to maturity such that \( \sigma_{1,t_0}(u) = \sigma_{2,t_1}(u) \),

(ii) the volatility function for each forward is constant and independent of other forwards such that \( \sigma_{1,t_1}(u) = \sigma_{2,t_1}(u) \),

(iii) the volatility function relies on the product of the time-remaining to maturity and a forward-rate-specific component and

(iv) the volatility function relies on the product of the time-bucket and forward-rate-specific component.

**Parameterisation**

The following parameterisation of instantaneous volatility has been suggested by Rebonato [Reb05] and Brigo et al. [BM07],

\[
\sigma_{\text{black},t_T} = (a + b\tau_i) \exp(-c\tau_i) + d. \quad (6.2.3)
\]

The form of (6.2.3) gives rise to the humped volatility smile in Figure 6.1 which is prevalent in empirical data.

![Figure 6.1: Rebonato Volatility Parameterisation with a = -0.05, b = 0.5, c = 1.5, d = 0.15](image)

Another parameterisation due to Svensson [Sve94] which is consistent with the Nelson and Siegel yield curve family is

\[
\sigma_{\text{black},t_T} = \beta_0 + (\beta_1 + \beta_2 t_i) \exp(-\alpha_1 t_i) + \beta_3 t_i \exp(-\alpha_2 t_i)
\]

where (i) \( \{\beta_0, \beta_1, \beta_2\} \) and (ii) \( \{\alpha_1, \alpha_2\} \) are free parameters that can be fitted to the volatility term structure.

We also assume a priori that the relationship

\[
\sigma_P(t,T) = \int_t^T \sigma_f(t,u) \, du + \text{constant.} \quad (6.2.4)
\]

holds between bond and instantaneous forward rate volatility functions.
6.3 Calibration

6.3.1 Introduction

A very important step in pricing products using any model is the calibration process which will identify candidate values for model parameters based on fitting the model to (i) the current yield curve and (ii) the current volatility term structure. Once a suitable volatility functional specification has been chosen as in section 6.2.2, the volatility function can be calibrated to the volatility term structure such that the stripped caplets and swaptions price correctly [BMM03].

In section 6.3.2 the paper reviews swaption approximation techniques. In section 6.3.3 the caplet formula is derived. This will be followed by a brief summary of rank reduction techniques used to reduce dimensionality after calibration (section 6.3.4).

6.3.2 Approximations of the swaption

The LIBOR market model calibrates easily to caplets quotes as can be seen in section 6.3.3, however there is no closed-form solution for swaptions prices. Several approximations for the swaption volatility have been proposed in the literature which help reduce calculation overhead in the calibration process (see Brigo et al. [BM07, p. 271] and Rebonato [JR03a]).

The LIBOR market model swaption approximations fall into one of two categories namely

(i) rank reduction techniques using Eckart-Young decomposition (theorem 6.3.4.2) as proposed by Brace [Bra96] or

(ii) freezing the drift terms swaption volatility approximations based on using time 0 values of the weights and/or forward rates.

The latter technique stems from the decomposition of a swap rate into a weighted function of the forward rates such that

\[ S_t(s,T) = \sum w_i(t) L_{T_t} \]  

(6.3.1)

where (i) \( w_i(t) \) is the simple weighting function at time \( t \). Using (6.3.1) we can calculate an approximation of the Black swaption volatility by integrating the percentage quadratic variation over the period that the swap rate is alive. The swaption volatility is then given by

\[ \int (d \log S_t(s,T)) (d \log S_t(s,T)) \]

The Rebonato drift approximation ([RR96] and [JR03a]) proposed that both the weights and forward rates are frozen at time 0. The approximation was found to reasonable by both Brace et al. [BDB01] and Brigo et al. [BM07].

A more sophisticated approximation by Hull and White [HW00] creates an adjusted weight function which is completely specified by the forward rates. The forward rates are then frozen at time 0 and thus by implication the adjusted weight function.

Other approximations were developed by Andersen and Andreasen [AA00], Daniluk and Gatarek [DG05] and Kawai [Kaw03] who used a Wiener chaos expansion. Brigo et al. discuss the techniques for approximating swaptions in the LIBOR market model in [BM07, p. 264]. Henrard [Hen09] also discusses approximation techniques in detail.

6.3.3 Caplet

The LIBOR market model prices caplets according to the Black-76 formula and calibration is thus simple. The theoretical derivation is given below. For details consult Brigo et al. [BM07, p. 202].
If $L_T^T$ satisfies
\[ \, dL_T^T = \sigma_L(t,T) L_T^T \, dW^{\mathbb{P}^{T+\delta}} (t) \]
under the forward measure $\mathbb{P}^{T+\delta}$ with initial condition at time 0 equal to $L_0^T$, then the solution of $L_T^T$ is given by the Doleans exponential (lemma E.0.6.3)
\[
\xi (L_T^T) = L_0^T \exp \left( \sigma_L(t,T) W - \frac{1}{2} \sigma_L(t,T)^2 T \right)
\]
where (i) the mean equals $m = -\frac{1}{2} \sigma_L(t,T)^2 T$, (ii) the variance equals $V^2 = \sigma_L(t,T)^2 T$ (by Ito’s isometry) and (iii) we have taken the volatility function $\sigma_L(t,T)$ to be in its simplest form which is a constant.

We can derive the price of a caplet fixed at $K$ by solving the risk neutral formulation (A.3.1)
\[
\pi (t) = \delta \mathbb{P}_t^{T+\delta} \cdot \mathbb{E}^{\mathbb{P}^{T+\delta}} \left( \max \left( \frac{L_T^T - K}{P_{T+\delta}^{T+\delta}} \right) | F (t) \right)
\]
\[
= \delta \mathbb{P}_t^{T+\delta} \cdot \mathbb{E}^{\mathbb{P}^{T+\delta}} \left( \max \left( L_T^T - K, 0 \right) | F (t) \right)
\]
\[
= \delta \mathbb{P}_t^{T+\delta} \cdot \int_{-\infty}^{\infty} \left( \max \left( L_T^T - K, 0 \right) \right) f (Y \in dy)
\]
\[
= \delta \mathbb{P}_t^{T+\delta} \cdot \int_{y} \left( L_0^T \exp \left( m + V \times y \right) - K \right) f (Y \in dy)
\]
\[
= \ldots
\]
\[
= \delta \mathbb{P}_t^{T+\delta} \left( L_0^T \Phi (d_1) - K \Phi (d_2) \right)
\]
(6.3.2)
where
- we integrate only from $\hat{y} = -\log \left( \frac{L_T^T}{K} \right) - m$ to infinity where $L_T^T > K$,
- $\Phi$ is the cumulative normal distribution for $X \sim \mathcal{N}(0,1)$,
- $\delta$ is the year fraction,
- $d_1 = \log \left( \frac{L_T^T}{K} \right) + \frac{1}{2} \sigma_L(t,T)^2 T$ and
- $d_2 = \log \left( \frac{L_T^T}{K} \right) - \frac{1}{2} \sigma_L(t,T)^2 T$.

As can be seen by (6.3.2), with the omission of the $\delta \mathbb{P}_t^{T+\delta}$ term, we have derived the Black-76 formulation which coincides nicely with the market volatility quotations.

6.3.4 Rank Reduction Techniques

Typical market models generally have one, two or three driving Brownian motions. Calibrating from historical or implied market variables will produce covariance matrices of rank larger than three which will need to be reduced via rank reduction techniques discussed briefly in this section.

Justification for using lower rank Brownian motion processes follows from numerous studies on the driving forces behind variation in prices. Analysis of the US government bond market by Litterman and
Scheinkman [LS91] identified three factors they referred to as (i) level, (ii) steepness and (iii) curvature which explained most of the variation in returns. Jamshidian and Zhu [JZ96, p. 45] determined that a single factor contributed between 68% and 76% of the total variation in returns in yield curves in Germany, Japan and the United States.

**Theorem 6.3.4.1** (Singular Value Decomposition). A $m \times n$ matrix $A$

\[
A = \begin{bmatrix}
    a_{1,1} & a_{1,2} & \ldots & a_{1,n} \\
    a_{2,1} & a_{2,2} & \ldots & a_{2,n} \\
    \vdots  & \vdots  & \ddots & \vdots  \\
    a_{m,1} & a_{m,2} & \ldots & a_{m,n}
\end{bmatrix}
\]

can be factored into

\[
A = U \Sigma V^T
\]

where (i) the columns of $U$ are the eigenvectors of $AA^T$, (ii) the columns of $V$ are the eigenvectors of $A^TA$ and (iii) the singular values (on the diagonal of $\Sigma$) are the square root of the eigenvalues (of $AA^T$ and $A^TA$). When $A$ is positive semi-definite, then $A$ simplifies to

\[
A = Q \Lambda Q^T. \tag{6.3.3}
\]

See Strang [Str03, p. 367] for more details.

**Theorem 6.3.4.2** (Eckart-Young decomposition). Assume matrix $A$ is specified as in (6.3.3) and that $\Lambda$ is sorted from largest to smallest, then the best performing $d$-rank reduction is $\hat{A} = Q \hat{\Lambda} Q^T$ where (i) $\hat{\Lambda} = \text{diagonal matrix } (\Lambda_1, \ldots, \Lambda_d, 0, \ldots, 0)$ and (ii) $\Lambda_1 > \Lambda_2 > \ldots > \Lambda_d$.

Practitioners have traditionally favoured normalised principal components analysis (PCA) as a rank reduction technique ([Reb05] and [BM07]). PCA borrows singular value decomposition (theorem 6.3.4.1) from linear algebra to factorise the covariance matrix into orthogonal components. Applying the Eckart-Young decomposition in theorem 6.3.4.2 to the correlation matrix $\hat{A}$ and normalising the decomposition (with matrix $M$) to get a unit valued diagonal, we get $\hat{A}M$ which is the new covariance matrix with rank equal to $d$.

**Definition 6.3.4.1** (Frobenius Norm). The Frobenius norm calculated over a matrix $\| A \|_F \{ A \subset \mathbb{R}^{m \times n} \} \to \mathbb{R}$ is given by

\[
\| A \|_F = \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{d} \hat{a}_{i,j}^2}
\]

In order to measure the effectiveness of any rank reduction technique we can apply the Frobenius norm in definition 6.3.4.1 on the matrix $A - \hat{A}M$ where $\| A_1 \|_F < \| A_2 \|_F$ would imply that rank reduction of $A_1$ was better than $A_2$.

Other rank-reduction techniques include (i) angles parameterisation [RJ11], (ii) alternating projections [Hig02], (iii) Lagrange multipliers [ZW03] and (iv) iterative majorisation technique [PG04]. For more details consult Selic [Sel06] or Brigo [BM07].

### 6.4 Simulating the LMM

#### 6.4.1 Introduction

In the LIBOR market model, Monte Carlo simulation involves the discretisation of the time variable $t$ into grid points $t \subset [0, \tau_1, \tau_2, \ldots, T^*]$ which includes all tenor dates (definition 2.2.1.1) applicable for the particular contingent claim. Part of the set up includes (i) choosing the form of $\sigma$ in section 6.2.2 (ii) choosing the measure under which the simulation will take place (martingale, spot
(definition 5.2.3.3 or forward), (iii) choosing the appropriate variable to simulate (e.g., log forward LIBOR or forward LIBOR) and (iv) choosing the discretisation scheme (e.g., Euler (definition 6.4.2.1)). Monte Carlo simulation is covered at length by [Gla03] and [Jac01]. [KP77] provide a good reference for discretisation techniques.

We reintroduce the arbitrage condition in terms of simulation in section 6.4.3, followed by a discussion on the predictor-corrector method used to reduce the number of time increments in section 6.4.4.

6.4.2 Discretisation

Definition 6.4.2.1 (Euler Scheme). Let $X$ be an Ito process satisfying

$$dX_t = a(t,X_t) \, dt + b(t,X_t) \, dW_t$$

with initial value $X_{t_0} = X_0$, then the Euler approximation over the interval $[t_0 \leq t \leq T]$ is

$$Y_{n+1} = Y_n + a(\tau_n,Y_n) (\tau_{n+1} - \tau_n) + b(\tau_n,Y_n) (W_{\tau_{n+1}} - W_{\tau_n}).$$

More details are available in Kloeden et al. [KP77].

Typical choices of discretisation schemes include (i) Euler (definition 6.4.2.1) or (ii) higher order schemes such as Milstein [Mil78] or Talay [Tal84]. The schemes need to ensure convergence and minimise error. In definition 6.4.2.2 we discretise the exponential LIBOR forward rate under the spot measure. For discretisation under the forward measure, the $u_n$ function will change.

Definition 6.4.2.2 (Log-Euler under the spot-measure). Discretise the forward LIBOR rate by

$$L_{\tau_{n+1}} = L_{\tau_n} \exp \left( \left( u_n (L_{\tau_n},\tau_n) - \frac{1}{2} \| \sigma_n (\tau_n) \|^2 \right) (\Delta t) + \sqrt{\Delta t} \sigma_n (\tau_n)^T Z_n \right)$$

where (i) $\Delta t = \tau_{n+1} - \tau_n$, (ii) $Z_n$ is a normal distribution with variance one and mean zero, (iii) $u_n$ is given by

$$u_n (L_{\tau_n},\tau_n) = \sum_{j=\eta(t)}^{\sum} \frac{\delta L_{\tau_n} \sigma_n (\tau_n)^T \sigma_j (\tau_n)}{1 + \delta L_{\tau_n}}$$

and (iv) $\eta(t)$ is right continuous and results in smaller discretisation error. More details are available in Glasserman [Gla03, p. 175].

6.4.3 No-arbitrage condition

Discretising continuous time models under a Monte Carlo simulation may cause the model to lose its no-arbitrage condition. The Musiela and Rutkowski weak arbitrage condition [MR97], under which the deflated zero coupon bonds $P_T^T\frac{P_T}{P_0}$ are martingales, needs to be met by the simulation. The choice of the drift term will dictate the discretisation behaviour of the model under the simulation. Glasserman [Gla03] states the root of the problem is choosing $u_n$ such that

$$E \left( \frac{1}{L_{\tau_{n+1}} (u_n) \ldots^{T}} | F_t \right) = \frac{1}{1 + \delta L_0}$$

will remain a martingale. Glasserman and Zhao [GZ00] provide a very detailed comparison on various Monte Carlo discretisation techniques for the LIBOR market model. Glasserman [Gla03] also is a good reference.
6.4.4 Predictor Corrector Model

Particular mention is made of the predictor-corrector modification to the Euler scheme, as it can be used in simulating the LIBOR market model with larger time steps, typically with time steps of size $\delta$ and intervals reflecting the tenor dates. We briefly define the method definition 6.4.4.1 due to Platen [KP77, p. 501].

**Definition 6.4.4.1 (Predictor Corrector).** *Predictor-corrector with predictor*

\[ \hat{Y}_{n+1} = Y_n + a(\tau_n, Y_n)(\tau_{n+1} - \tau_n) + b(\tau_n, Y_n)(W_{\tau_{n+1}} - W_{\tau_n}) \]

*and corrector*

\[ Y_{n+1} = Y_n + \frac{1}{2} \left( a(\tau_n, Y_n) + a(\tau_n, \hat{Y}_{n+1}) \right)(\tau_{n+1} - \tau_n) + b(\tau_n, Y_n)(W_{\tau_{n+1}} - W_{\tau_n}) \]

Application of predictor-corrector can be found in Kurban et al. [KSS02] and Hunter [HJJ01].

6.4.5 Finite Difference Methods

**Definition 6.4.5.1 (Separable Volatility Functions).** *A collection of volatility functions $\sigma_i(t) : \{t \subset [0,t_i]\} \to \mathbb{R}^d$ are separable if there exists a function $\sigma(t) : \{t \subset [0,t_i]\} \to \mathbb{R}^d$ and vector $v_i \to \mathbb{R}^d$ such that $\sigma_i(t) = v_i \sigma(t)$.*

Work by Pietersz [PPVR04] describe the conditions of separable volatility in definition 6.4.5.1 under which the LIBOR market model can be calculated under finite difference schemes.
Chapter 7

Smile Modelling

7.1 Introduction

Caplet and swaption volatility quotes demonstrate both a time and strike dimension resulting in a volatility smile effect. Market models can accommodate the term structure of volatility through a suitable choice of volatility function (section 6.2.2), however modifications to the model are required to incorporate the variation in volatility quotes due to strike \[ \text{RW09, p. 51} \].

The economic rationale \[ \text{Jac04} \] of the smile is particular to the market (FX, Equity, Interest Rates). In equity markets, the skew is explained by (i) demand and supply differences or (ii) leverage effects \[ \text{Rub83} \] as discussed in section B.1, while in the interest rate markets the skew could be symptomatic of the (i) expectation of rate levels driven by the central banks as a result of monetary policy objectives and (ii) rates reversion properties and the natural boundary conditions rates travel in \[ \text{2} \].

Several general methods for adding smile dynamics to the LIBOR market model are summarised in Meister \[ \text{MF04} \], Brigo et al. \[ \text{BM07, p. 451} \] and \[ \text{AP10b} \] including (i) local volatility models (section B.1), (ii) models with jump processes (section B.2), (iii) stochastic volatility models and (iv) others (such as uncertain parameter models, levy-driven models, market models of implied volatility, discrete time stochastic volatility models). Zhu \[ \text{Zhu07} \] suggest two general categories namely (i) local volatility modelling and (ii) stochastic modelling.

Meister \[ \text{MF04, p. 32} \] proposes an evaluation criteria for LIBOR models with a smile dimension namely (i) model parameters should be meaningful and stable, (ii) the volatility smile shape should remain time homogeneous (maintain its shape) and (iii) the model should calibrate to all tenors and strikes. Papapantoleon \[ \text{Pap10} \] suggests that models would need to exhibit properties of tractability, positive forward rates and arbitrage-free dynamics. Rebonato expands on the idea of tractability in \[ \text{Reb08} \] and includes the idea that models need to calibrate without Monte Carlo simulation. Henry-Labord’ere \[ \text{HL07} \] discusses the calibration issue as well.

Only the application of stochastic volatility is considered in this chapter. In section 7.2 we commence with a general discussion on smile modelling within a Black-Scholes framework. We then provide some analysis on stochastic volatility smile modelling in section 7.3 and follow this with a discussion on the LLM-SABR extension in section 7.4. The appendix section B provides a brief introduction to local volatility, the Constant Elasticity of Variance (CEV) property and jump models.

\[ ^1 \text{Rebonato speaks about hockey stick smiles requiring model adjustment.} \]

\[ ^2 \text{Rates are generally always positive. They are higher in good economic times with high inflation and lower in recessions. This automatically places bounds on the rates.} \]
7.2  The Implied Volatility Function

When the market implied volatility is no longer constant, this affects both the distribution properties of the underlying as well as the hedging characteristics of the model. These topics are covered in section 7.2.1 and section 7.2.2 respectively.

7.2.1  Distribution Properties

Under the risk neutral measure the stock price in the Black-Scholes model follows a log-normal distribution such that

\[ S_T \sim S_0 e^{m + V \times N(0,1)} \]  

(7.2.1)

where (i) the mean equals \( m = -\frac{1}{2} \sigma^2 T \), (ii) the variance equals \( V^2 = \sigma^2 \times T \) with \( \sigma \) constant and (iii) \( N(0,1) \) is the standard normal distribution.

Because the Black-Scholes constant volatility assumption was violated by markets quoting different implied volatilities for each strike, it became necessary to calculate the risk neutral distribution of the stock price which incorporated the market quotes. Breeden and Litzenberger [BL78] established a relationship between the call price and the risk neutral probability distribution for \( S_T \) such that

\[ f_t(S(T) \in dK) = \mathbb{E}_t(\delta(S(T) - K)) = \frac{\partial^2 c(t,S(t);T,K)}{\partial K^2}, \]  

(7.2.2)

where (i) \( f \) is the probability function, (ii) \( \delta \) is the Dirac delta function and (iii) \( c \) is the un-discounted price of the call. For the second derivative to exist in (7.2.2) there would need to be a continuum of strikes and option prices.

7.2.2  Backbone and Smile Dynamics

An important factor in evaluating an extended LIBOR market model is how realistic its dynamics are to underlying stock movements. Piterbarg and Andreasen [AP10a, p. 699] state that the value of a model to a trading desk is not the ability to fit to the market, but rather how the model dynamics match the market. Local volatility model dynamics display an incongruity with empirical evidence as they suggest incorrect dynamics to underlying changes [HKLW02].

To make the idea about smile dynamics more concrete, the Black-Scholes call option formula can be rewritten in terms of the implied volatility function \( iv_B(S,t,T,K) \) such that

\[ c_B(iv_B(S,t,T,K),T,S,r,K,t). \]  

(7.2.3)

Differentiating (7.2.3) with respect to \( S \) using the chain rule gives

\[ \Delta_{iv}(t) = \frac{\partial c_B}{\partial S} + \frac{\partial c_B}{\partial iv_B} \frac{\partial iv_B}{\partial S} = \Delta + \Upsilon \frac{\partial iv_B}{\partial S} \]  

(7.2.4)

where (i) \( \Delta \) is the Black-Scholes delta, (ii) \( \Upsilon \) is the Black-Scholes Vega, (iii) \( \Delta_{iv} \) is the modified delta using the chain rule, (iv) \( iv_B \) is the implied volatility function and (v) \( c_B \) is the call price of a European option. The modified delta \( \Delta_{iv} \) in (7.2.4) contains an additional term which includes the Black-Scholes Vega multiplied by the \( \frac{\partial iv_B}{\partial S} \) term. The term \( \frac{\partial iv_B}{\partial S} \) is sometimes referred to as the volatility backbone.

The requirements that smile dynamics reflect the market is so important to traders that they will sometimes override the default behaviour of the backbone by specifying rules on how the smile dynamics behave. This is called shadow delta hedging and typical rules for smile dynamics include (i) a sticky strike rule which assumes dynamics remain unchanged in response to price moves and (ii) a sticky delta rule which assumes the smile simply moves unchanged in shape to a new ATM anchor point.
7.3 Stochastic Volatility

Stochastic volatility, which is based on rectifying the empirically unjustifiable assumption of a constant market observable volatility, has remained a popular method of introducing smile dynamics. In the case where the stochastic volatility process is uncorrelated with that of the underlying, the smile is parabolic or symmetrical in nature.

Stochastic volatility analysed under a risk neutral framework introduces its own problems notwithstanding the obvious issues of introducing an exogenous market price of volatility. Zhu expands on issues that need particular attention under stochastic volatility Market Models namely (i) correlation between the underlying and variance process need careful attention under the change of numéraire, (ii) there is an abundance of model parameters needing calibration which are resultant from multiple tenors in term structure models and (iii) the requirement for approximations for the swaption to support the calibration process. Lewis [Lew12] discusses the breakdown of stochastic volatility models under certain scenarios where there is a possibility of an explosion of volatility in the axillary or risk-adjusted processes.

In section 7.3.1 the paper introduces the reader to the basic form of the stochastic volatility model. This is followed by mentioning some of the significant contributions to the field of stochastic volatility modelling (section 7.3.2) including protagonists within the Market Models. In section 7.3.3 the paper analyses the smile within a stochastic volatility framework. It concludes with a discussion on the Hull and White no-arbitrage PDE in section 7.3.4. To better understand stochastic volatility the reader is referred to an excellent but challenging reference by Lewis [Lew12].

7.3.1 The Basic Stochastic Volatility Model

The basic idea of stochastic volatility is to relax the assumption around constant variance made in Black-Scholes and to model the variance process as another SDE. The stock price now satisfies the SDE

\[ dS_t = (\alpha_t S_t - D_t) \, dt + \theta(S_t) \sigma_t S_t dW_t \]  

(7.3.1)

with the variance process satisfying

\[ dV_t = b(V_t) \, dt + \xi \eta(V_t) \, dZ_t \]  

(7.3.2)

where

\[ V_t = \sigma_t^2. \]

Here (i) \(D_t\) is the dividend rate, (ii) \(\alpha_t\) is the instantaneous stock return, (iii) \(W\) and \(Z\) are two Brownian motions having correlation \(p(V_t)\), (iv) \(b\) is the real world drift of the variance (this can be modelled as an Ornstein–Uhlenbeck process), (v) \(\xi\) is the volatility of volatility and (vi) \(\eta(V_t)\) and \(\theta(S_t)\) introduce CEV properties.

The generalised stochastic volatility model (SV) presented above is more complicated than the Black-Scholes formulation and importantly in the absence of tradable volatility products is incomplete. The market price of volatility must therefore be exogenously supplied. There are many variants of the SV model above (i) such as where the correlation between the two Brownian motions is 0, (ii) where CEV (section B.1.1) attributes are introduced to the underlying \(V_t\) as in the SABR model, (iii) and where CEV properties are introduced into the variance process by modifying \(\eta(V_t) = V_t^m\).

7.3.2 Background

Empirical evidence such as studies by MacBeth and Merville [MM79] and Rubinstein [Rub85] have rejected constant volatility.
The problem of modelling heterogeneous variance is not a new one. In 1973, Clark [Cla73] used a subordinated stochastic process based on a time change to explain the large kurtosis of the empirical stock price distributions.

In 1987 Hull and White [HW87] introduced a model that had a stochastic variance process which was uncorrelated to that of the stock. Their paper derived a PDE based on work by Garman [Gar76] which would hold if the model admitted no-arbitrage (section 7.3.4). Other developments in the same year included Scott [Sco87], Wiggins [Wig87] and Johnson and Shannon [JS87].

In 1993 Heston [Hes93] introduced a correlated stochastic volatility model that admitted a closed form solution using Fourier transformations. Its tractability made it a popular stochastic volatility model used in Forex and other markets. Further studies included Stein and Stein’s work on Fourier inversion [SS91] and Ball and Roma [BR94] comparison of Fourier inversion and power series techniques.

Later in 2002, Hagan et al. [HKLW02] proposed a CEV (section B.1.1) model with asymptotic approximations which has found acceptance in the interest rate world. Several LIBOR market model extensions based on stochastic volatility have been proposed namely

(i) Andersen and Andreasen [And02] who extended the Heston model and used Rebonato freezing technique to approximate swaptions,

(ii) Piterbarg [Pit03] who extended the Andersen and Andreasen to extend over the whole term structure,

(iii) Henry-Labord’eere [HL07], Hagan and Lesniewski [HL08], Morini and Mecurio [MM07], Rebonato [Reb07], Rebonato and White [R09] and Rebonato, White and McKay [RMW11] who applied SABR dynamics to market models,

(iv) Andersen and Brotherton-Ratcliffe [ABR05],

(v) Joshi and Rebonato [JR03b],

(vi) Zhu [Zhu07],

(vii) and Wu and Zang [WZ06] who build their model on the Heston and derive swaptions approximations.

Shepard and Andersen [SA09] provide a literature overview of modelling stochastic volatility from Clark’s (1973) mixture models approach.

7.3.3 The Smile under Stochastic Volatility

In this section we investigate exactly how the addition of stochastic volatility supports the smile shape. We first discuss the asymptotic expansion of the implied volatility function followed by the powerful concept of ‘ mixing solutions ’.

Assymptotic Expansion of the Implied Volatility Function

Under a stochastic model the implied volatility function can be expanded around the volatility of volatility parameter $\xi$. The first step is to perform a generalised Fourier transform of the option pricing formula. The option price can then be backed out by integrating the transformed function over the complex k-plane. The k-plane integrals can be done analytically because they reduce to derivatives of the Black-Scholes formula. Lewis [Lew12, p. 76] provides a complete and rather comprehensive

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Lewis [Lew12] p. 36 discusses the ‘fundamental transform’ of the option price which involves the use of a Fourier transform of the payoff function.
The analysis of the expanded implied volatility function provides additional insight into the shape of the smile. When correlation between the Brownian motions in equations (7.3.1) and (7.3.2) is zero, the smile is symmetric. Using the expansion idea above, Lewis [Lew12, p. 129] demonstrates that when the correlation is zero, the implied volatility function is approximated by a quadratic function which is symmetrical around the time average volatility. He similarly applies the mixing solution discussed in section 7.3.3 to show how the implied volatility is again quadratic.

Mixing Solutions

The idea that stochastic volatility model calls and puts could be represented by a weighted sum of constant volatility Black-Scholes prices was first demonstrated by Hull and White [HW87] for an uncorrelated variance process. Later Romano and Touzi [RT97] proved this for a correlated processes. Lewis [Lew12, p. 99] refers to this as the mixing idea and gives examples of how the mixing theorems can be applied to stochastic volatility problems namely (i) improving Monte Carlo simulations, (ii) providing analytical solutions for the mixing steps and (iii) providing series expansion for volatility of volatility (section 7.3.3).

The mixing idea is a particularly powerful result as it expresses more complicated models such as stochastic volatility in terms of simpler models like Black-Scholes. Following Lewis, the first result is that the stock price can be treated as an evolution of a stock price under constant deterministic volatility plus an adjustment based on the stochastic volatility. The second result is that the Black-Scholes formula can be used to calculate the stochastic volatility price by replacing the stock price with an effective stock price and the volatility with an effective volatility.

7.3.4 Hull and White No-arbitrage Condition

Definition 7.3.4.1 (Black-Scholes Real World SDE). Black-Scholes [BS73] proposed that the stock price $S$ satisfied

$$dS = S\left(\mu dt + \sigma\hat{W}(t)\right) \tag{7.3.3}$$

under the real world measure.

In 1987 Hull and White [HW87] proposed that the variance process $\mathcal{V}_t$ satisfied the following SDE

$$d\mathcal{V}_t = \mathcal{V}_t\left(\alpha_v(\sigma,t)dt + \sigma_v(\sigma,t)\hat{W}\right) \tag{7.3.4}$$

under the real world measure where (i) $\mathcal{V}_t = \sigma^2$, (ii) the Brownian motions $\hat{W}$ and $W^P(t)$ were correlated by $\rho$ and (iii) was specified as above.

Definition 7.3.4.2 (Black-Scholes Risk Neutral SDE). Apply the Girsanov process with market price of risk equal to $\theta(t) = \frac{\mu - r}{\sigma}$ we get

$$dS = S\left(rd\mu + \sigma W^Q(t)\right) \tag{7.3.5}$$

under the risk neutral measure.

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4Lewis derives these results for various CEV flavours of the stochastic models namely where $\eta(V_t) = V_t^m$ for various values of $m$.

5This idea is again applied in the LLM-SABR analysis when looking at the backbone of ATM volatilities.

6The interested reader can refer to Lewis [Lew12, p. 99] for details.
Definition 7.3.4.3 (Garman’s PDE for state variables.). Garman proved that a security $v$ which depended on state variables $\gamma_i$ (such as stock price, volatility etc) must satisfy

$$\frac{\partial f}{\partial t} + \frac{1}{2} \sum_{i,j} \rho_{i,j} \sigma_i \sigma_j \frac{\partial^2 f}{\partial \gamma_i \partial \gamma_j} - rf = \sum_i \gamma_i \frac{\partial f}{\partial \gamma_i} (-\alpha_i + \beta_i (\alpha_* - r))$$

(7.3.6)

where (i) $\gamma_i$ is state variable $i$, (ii) $\rho_{i,j}$ is the correlation between state variables, (iii) $\sigma_i$ is the standard deviation of variable $i$, (iv) $\alpha_i$ is the drift of variable $i$, (v) $r$ is the risk free rate, (vi) $\alpha_*$ is the expected returns of the market and (vii) $\beta_i$ is the regression coefficient of the particular state variable returns ($\frac{\partial \gamma_i}{\partial \gamma}$) to the market. The work by Garman [Gar76] related the Capital Asset Pricing Model (CAPM) to PDEs.

Hull and White [MR05, p. 256] applied simplifying assumptions to definition 7.3.4.3 namely (i) volatility was uncorrelated to market returns (i.e. $\beta_V = 0$) (ii) and market price of risk for stocks was specified as in definition 7.3.4.2.

They arrived at the simplified PDE

$$\frac{\partial f}{\partial t} + \frac{1}{2} \left( \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + 2 \rho \sigma^3 \alpha_v (\sigma,t) S \frac{\partial^2 f}{\partial S \partial V_t} + V_t^2 \sigma_v^2 (\sigma,t)^2 \frac{\partial^2 f}{\partial V_t^2} \right)$$

$$-rf + rS \frac{\partial f}{\partial S} + a \sigma^2 \frac{\partial f}{\partial V_t} = 0$$

(7.3.7)

which the price of a contingent claim $f$ would need to satisfy under stochastic volatility. In general the model was dependent on an exogenous specification of the market price of volatility (i.e. $\lambda(t)$) and not complete unless volatility was tradable.  

7.4 LLM-SABR

Although not a term structure model, the SABR model developed by Hagan et al. [HKLW02] gained recognition due to its ability to model the volatility smile. It provided a viable alternative to the local volatility models developed by Dupire [Dup94] and Derman et al. [DKZ96] which predict incorrect smile dynamics [Hau07]. The asymptotic approximations of calls and puts under the SABR made it particularly appealing. Additionally, it was tractable and easy to calibrate to market volatilities.

SABR’s success in interest rate environments has been pivotal in contributing to the publishing of several LIBOR market model extensions based on SABR. The SABR model is not without its problems. In low interest rate environments in which most of the world found itself after the global financial crisis SABR struggles to calibrate. Zhu [Zhu07] believes there are still open issues surrounding the application of SABR to LMM in the original paper by Hagan et al. [HKLW02] namely (i) mean reversion properties are not modelled which is in contradiction to empirically evidence, (ii) swaption and caplet models still have no mathematical link, (iii) the volatility process still remains non-stationary and (iv) the changes in the variance process was not clear under the forward measure.

Henry-Labordere applied the SABR technique to the LIBOR market model [HL07]. Here he presents an interpretation on LLM-SABR based on the expansion of the heat kernel in hyperbolic geometry using Riemannian manifolds. He discards the freezing of drifts to replace it with the derivation of an asymptotic local volatility which he uses to derive an approximation to the swaption price. The author’s work shows that for short dated options, his expansion is incompatible with the freezing argument originally suggested by Rebonato.

Mecurio and Morini [MM07] develop a model based on SABR while demonstrating that the no-arbitrage conditions still hold. The authors initially begin with the SABR dynamics applied to each

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7 This is true because we still have the $\alpha_v (\sigma,t)$ term.  
8 Hagan is infamous for among other things, the London Whale incident in which JP Morgan lost large amounts of money which was being managed by a new VAR model produced by Hagan.
forward rate. The authors then apply the change of measure while safeguarding the no-arbitrage condition. Additionally they develop approximations in line with those developed under SABR while preserving the single stochastic volatility factor and constant parameters of the original SABR model. Rebonato et al. [RMWT] also apply SABR dynamics to the Market Model. Unlike Mecurio and Morini, here the authors use the LIBOR market model as a base with which to replicate the dynamics of the SABR model. The authors parameterisation remains time-homogeneous and stable.

Although there are several variations based on SABR, we follow the paper by Hagan and Lesniewski [HL08] and augment this with analysis done by Brigo et al. in [BM07]. In the absence of tradable volatility products, a stochastic model with a drift term in the variance process is incomplete and requires exogenous specification of the market price of volatility. It should be noted that the LLM-SABR is convenient in that it simplifies the variance process by assuming zero drift.

In section 7.4.1 we present the LLM-SABR dynamics, followed by a discussion on the model in section 7.4.2. In section 7.4.3 we discuss the expansion of the implied volatility function.

7.4.1 Dynamics

Hagan and Lesniewski [HL08] combine SABR dynamics with the LIBOR model such that under the k-forward measure the simple LIBOR rate \( L_{i,t} \) satisfies

\[
dL_{i,t} = \Delta_j (t) \, dt + \sigma_j (t) \, L_{i,t}^{\beta_j} \, dW_j (t) \tag{7.4.1}
\]

with the variance process given by

\[
d\sigma_j (t) = \xi_j (t) \, \sigma_j (t) \, \Lambda_j (t) \, dt + \xi_j (t) \, \sigma_j (t) \, dZ_j \tag{7.4.2}
\]

where (i) \( \Delta_j (t) \) is the drift of \( L_{i,t} \) under the k-forward measure, (ii) \( \Lambda_j (t) \) is the drift of the variance process under the k-forward measure, (iii) \( \xi_j (t) \) is the stochastic volatility of the \( L_{i,t} \) forward, (iv) \( \beta_j \) is the CEV parameter for the \( L_{i,t} \) forward and (v) \( W_j \) and \( W_j \) are the Brownian motions. The correlations between the Brownian motion processes \( W \) and \( Z \) are given by

\[
\begin{align*}
E (dW_j dW_k) &= \rho_{j,k} dt \\
E (dW_j dZ_k) &= r_{j,k} dt \\
E (dZ_j dZ_k) &= \eta_{j,k} dt
\end{align*}
\]

with the matrix

\[
\Pi = \begin{bmatrix} \rho_{j,k} & r_{j,k} \\ r_{j,k}^\top & \eta_{j,k} \end{bmatrix} \tag{7.4.3}
\]

assumed to be positive definite.

The forward rate drift is calculated using a standard induction technique used by both Jamshidian and Brace et al. for the standard LIBOR market model (proof 5.2.2). Here the drift is given by

\[
\Delta_j (t) = \begin{cases} 
-\sigma_j (t) \, L_{i,t}^{\beta_j} \, \sum_{j+1 \leq i \leq k} \frac{\rho_{j,i} \, \delta \, \sigma_j (t) \, L_{i,t}^{\beta_j}}{1 + \delta \, L_{i,t}^{\beta_j}} & \text{for } j < k \\
\sigma_j (t) \, L_{i,t}^{\beta_j} \, \sum_{k+1 \leq i \leq j} \frac{\rho_{j,i} \, \delta \, \sigma_j (t) \, L_{i,t}^{\beta_j}}{1 + \delta \, L_{i,t}^{\beta_j}} & \text{for } j > k \\
0 & \text{for } j = k.
\end{cases} \tag{7.4.4}
\]

The stochastic process drift is given by

\[
\Lambda_j (t) = \begin{cases} 
-\sigma_j (t) \, L_{i,t}^{\beta_j} \, \sum_{j+1 \leq i \leq k} \frac{r_{j,i} \, \delta \, \sigma_j (t) \, L_{i,t}^{\beta_j}}{1 + \delta \, L_{i,t}^{\beta_j}} & \text{for } j < k \\
\sigma_j (t) \, L_{i,t}^{\beta_j} \, \sum_{k+1 \leq i \leq j} \frac{r_{j,i} \, \delta \, \sigma_j (t) \, L_{i,t}^{\beta_j}}{1 + \delta \, L_{i,t}^{\beta_j}} & \text{for } j > k \\
0 & \text{for } j = k.
\end{cases} \tag{7.4.5}
\]
The model is initialised with the forward rates derived directly from the yield curve such that $L_{t,t_j} = F_j (0)$ and is specified for (i) the spot measure, (ii) the k-forward measure (iii) and the swap numéraire.

### 7.4.2 LLM-SABR

The LLM-SABR model is over determined meaning that several good fits can be obtained to the market. Hagan et al. and Rebonato [RW09, p. 40] both suggest the a-priori choice of $\beta$ (usually $\beta = 0.5$). Brigo suggest an alternative historical regression technique in [BM07, p. 510]. Rebonato [RW09] states that beta should be chosen so as to make the volatility of volatility and correlation as stable as possible. He further goes on to investigate empirical evidence for the value of $\beta$.

When the beta coefficient is determined by calibration, beta is calibrated against each caplet forward rate and not to the swaptions. The betas for caplets and swaptions have no simple relationship but an asymptotic relationships can be derived [BM07]. Investigation by Jourdain [Jou04] proved that for $\beta < 1$, the forward process remains a martingale. With the log-normal case of $\beta = 1$ there are restrictions placed on the correlation coefficient such that $\rho_{j,k} \leq 0$.

The shape of the smile is controlled by $r_{j,k}$ and $\eta_{j,k}$, with the spot smile controlled predominantly by $r_{j,j}$ and the forward smile controlled by principally by $\eta_{j,k}$. There exists no explicit solution for the model under the existence of a smile, however when the correlations are zero, an explicit solution exists.

### 7.4.3 Expansion

The use of low noise expansion (see [FW12] and [KP77]) are used by the authors to derive an approximation which supports quick calibration. Additionally, the single perturbation techniques provide some insight into contribution of each parameter to the volatility smile.

Using singular perturbation techniques around the expansion variable $\xi_j (t)$ Hagan et al. [HL08] derive a formula for the implied volatility

$$
\sigma^{imp} (K,F) = \left( \frac{1}{1 + \frac{(1-\beta)^2}{24} \ln \left( \frac{F}{K} \right)^2 + \frac{(1-\beta)^4}{1920} \ln \left( \frac{F}{K} \right)^4 + \ldots} \right) \frac{z}{x (z)}
$$

where

(i) the equation is parameterised by the strike $K$ and forward rate $F = L_{t,t_j}$,

(ii) $z$ is given by

$$
z = \frac{\xi_j (t)}{\alpha} (FK)^{\frac{1-\beta}{2}} \ln \left( \frac{F}{K} \right),
$$

(iii) $x$ is given by

$$
x (x) = \ln \left( \frac{\sqrt{1 - 2\rho_{j,k} z + z^2 + z - \rho_{j,k}}}{1 - \rho_{j,k}} \right),
$$

(iv) the variance at time 0 is $V (0) = \alpha$,

(v) $\beta$ is the [CEV](https://en.wikipedia.org/wiki/Constant_elasticity_of_variance) parameter for the forward rate and

(vi) $\xi_j (t)$ is the volatility of volatility.

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9See Brigo and Mecurio [BM07, p. 509] for details.
The implied volatility in (7.4.6) can then be substituted into the standard caplet formula derived earlier in (6.3.2).

To get the ATM backbone (see section 7.2.2) implied volatilities, \( K = F_j(0) \) can be substituted into (7.4.6) to get

\[
\sigma^{imp}(F,F) = \frac{\alpha}{F_j(0)^{1-\beta}} (\ldots) \tag{7.4.9}
\]

where the leading term in (7.4.9) is a function of both \( \alpha \) and \( \beta \).

Hagan et al. [HKLW02] use an approximation of the implied volatility based on expansion of \( \log \left( \frac{K}{L_r} \right) \) to investigate the meaning of non ATM strikes. The implied volatility contributions can be broken down into

(i) the ATM contribution

(ii) the components of \( \log \left( \frac{K}{L_r} \right) \) which are (a) ‘beta’ skew term \( \frac{1}{2}(1 - \beta) \) and (b) ‘Vanna’ skew term \( \frac{1}{2}\lambda\rho \)

(iii) the components of \( \log \left( \frac{K}{L_r} \right)^2 \) which are (a) ‘beta’ skew squared term and (b) ‘Volga’ skew term \( \lambda^2 \)
Chapter 8

Conclusion

The paper has presented the development of interest rate modelling from the earliest first short rate model by Vasicek, through to HJM, followed by BGM and finally LLM-SABR. Evident from this chronology is that interest rate modelling seems to have followed a natural and incremental progression in sophistication and market fit. (i) HJM removed the restriction on having a single short rate and replaced this with a term structure which fitted the yield curve. (ii) Market models moved from modelling unobservable interest rates under HJM to observable LIBOR rates. (iii) The SABR model went on to model a single underlying under a heterogeneous variance. (iv) LLM-SABR extended this to incorporated the full term structure with their interrelationships into a single model.

What has been evident from this paper is that a large driver in the acceptance and take up of a model has remained its ability to calibrate efficiently to market prices. This is really mirrored by the adoption in the financial community for models: (i) where closed form solutions exist, (ii) approximation techniques are available, or (iii) when models reduce to simpler forms at calibration nodes. The paper has reviewed some of these approximation techniques under sections on (i) calibration (section 6.3) and stochastic volatility. Approximation methods such as (i) mixing techniques used by Clark, Lewis etc.; (ii) Fourier integrals used by Heston, Lewis, etc.; (iii) freezing used by Rebonato etc.; (iv) manifold theory used by Henry-Labordère and (v) power expansions used by Hagan etc., become more important as the models become more complicated.

A theme throughout the paper has been the no-arbitrage framework based on the relative pricing first proposed by Black-Scholes. This theme is so important that the paper devotes an appendix (appendix A) to the development of this work by Kreps, Harrison, Pliska and others.

(i) In the short rate models the no-arbitrage analysis in section 3.4 leads us to a PDE which a contingent claim would need to satisfy in order to remain arbitrage free. It also demonstrates that under the no-arbitrage analysis the short rate models are incomplete without the exogenous specification of a yield curve.

(ii) In the HJM model, the authors develop a no-arbitrage relationship between the drift and volatility parameters.

(iii) No-arbitrage analysis of the market models showed that under the k-forward measure, the LIBOR rates would need to contain a drift in order to remain arbitrage free. This was proved through induction by Brace et al. and Jamshidian.

(iv) LLM-SABR derived drift terms for both the underlying LIBOR rate as well as the variance process.

---

1Black 76 is used for caplets, floorlets and swaptions.
2SABR is used for European interest rate derivatives and SABR parameters can be used by the market to describe the smile.
3Market models reduce to the Black '76 formula at node dates.
(v) Even during practical implementation, the no-arbitrage analysis is in the forefront. The paper describes the work of Glasserman and Zhao (section 6.4.3) who apply the no-arbitrage principal in evaluating the different simulation options. Additionally the no-arbitrage principal gets used to improve simulation in the predictor corrector model by Kloeden etc..

There still remains open issues around LLM-SABR namely (i) it does not reflect the mean reverting nature of the variance process, (ii) nor does it explain or rectify the incompatibility between LLM-SABR and SABR/LSM and (iii) as emphasised by Zhu [Zhu07] and Mecurio and Morini [MM07] there remains more work to be done on how no-arbitrage can be maintained during change of measure under correlated Brownian motions.

Often the question is posed whether there exists a single all encompassing model. Yet this paper would argue that the question of how to price interest rate derivatives correctly still remains open to some degree. The basic tenet of all modelling as demonstrated above has remained the no-arbitrage principal. The ability to (i) hedge in continuous time, (ii) with no spread or transaction costs, (iii) in an efficient market that reflects all information can be argued to be fundamentally flawed. It is more likely that not all market participants want to price under market efficiency as it reduces profit and margins. It is also likely that economic forces cannot be represented by simple mathematical constructs, that models will always need to evolve to reflect changing market attributes and that individual economic rationales do not always follow traditional logic and decisions maybe have some element of game theory logic. But without the existence of an alternative, the risk neutral framework provides a viable, albeit approximate method for pricing contingent claims.\footnote{The markets are certainly being forced to move toward more transparency through regulation such as Basel and Dodd-Frank. This leads to lower spreads, larger trading windows, reduced risk, more transparency and brings the market more in line with theoretical underpinnings of risk neutral pricing.}
Appendix A

No Arbitrage Pricing

A.1 Introduction

In this section we construct a theoretical model under which we may price contingent claims. We wish to construct the model to theoretically justify the no-arbitrage basis under which our interest rate contingent claims are priced. Our treatment is not rigorous, but the interested reader may consult Musiela et al. [MR97, p. 279], whom we follow closely. The original papers by Pliska [Pli97], Harrison [HK79] and [HP81] or textbooks by Shreve [Shr], Nefci [Nef00] or Bass [Bas03] are further good references.

A.2 Risk Neutral Pricing Framework

The valuation of contingent claims under a risk neutral framework borrows concepts such as measure theory, martingales and calculus of functions of infinite variation from mathematical analysis. The end goal of the application of the mathematical rigour, is to relate prices of a replicating portfolio to that of the contingent claim via arguments of no-arbitrage.

The Black-Scholes 1973 paper [BS73] laid the groundwork for this risk neutral valuation, but it was however only later that Pliska et al. laid down a more rigorous framework that defined conditions under which a predictable self-financing trading strategy could be used to replicate an attainable contingent. The key result was that the spot value of the replicating trading strategy normalised over a numéraire would be equal to the price of the contingent claim today via arguments of no-arbitrage.

A.2.1 Theoretical Economy

We begin with a theoretical economy defined over the time interval \( t \subset [0,T] \) for some fixed \( T > 0 \), in which investors hold long or short positions in \( k \) tradable assets.

**Definition A.2.1.1 (Filtration).** The filtration \( \mathcal{F}_t \) is the set of realisations of the price process up to and including \( t \) generated by the price process \( Z_t \). It satisfies the usual conditions.

We introduce uncertainty into the economy via a probability space \((\Omega, F, \mathbb{P})\), whereby events unfold over time according to the filtration \( \mathcal{F}_t \) definition A.2.1.1. At each time period \( t \) the filtration permits market participants access to past prices. The probability \( \mathbb{P} \subset \mathcal{P} \) is only one of many equivalent measures held by individuals within the economy.

**Definition A.2.1.2 (Price Process).** A price process is a stochastic process \( Z_t = (Z_1, \ldots, Z_k) \) defined by a \( R^k \) vector of prices for each tradable asset at time \( t \). The price process is assumed to be a
decomposable semi-martingale, such that \( Z(t) = Z(0) + \int_0^t \mu(s, \omega) \, ds + \int_0^t \sigma(s, \omega) \, dW(s) \). Conditions apply on both the \( \mu \) and \( \sigma \) functions that they be adapted to the filtration \( \mathcal{F}_t \) and also that they exist in \( L^1 \) and \( L^2 \) respectively.\(^1\) The Ito integral can only be guaranteed to be a local martingale under the \( L^2 \) condition.

The price outcomes in the economy are uncertain and the price process \( Z_t \) definition A.2.1.2 is assumed to follow a positive semi-martingale process.

**Definition A.2.1.3** (Trading Strategy). A trading strategy is a predictable stochastic process \( \phi_t = (\phi_1, \ldots, \phi_k) \) defined by the \( \mathbb{R}^k \) vector of holdings in each tradable asset at time \( t \). A trading strategy is self-financing if the change in the wealth process definition A.2.1.4 is purely from capital gains.

We assume individuals participate in trading strategies definition A.2.1.3. The trading strategies are frictionless and occur no transaction costs nor do they restrict short selling. The price process is also continuous, which implies that participants may trade on all prices on a stochastic price path.

**Definition A.2.1.4** (Wealth Process). The value of the trading strategy \( \phi_t \) at time \( t \) is \( V_t = Z_t \cdot \phi_t \) where \( \cdot \) is the dot product of the two vectors.

**Definition A.2.1.5** (Gains Process). The gains process \( G_t = V_t - V_0 \) is the change in the value of the trading strategy between time \( 0 \) and \( t \). Mathematically we construct an Ito integral such that \( G_t = \int_0^t \phi_u dZ_u \). The gains process is defined purely in terms of the changes in the price process and is therefore self-financing.

**Definition A.2.1.6** (Predicable Process). A predictable process is adapted to the filtration \( \mathcal{F}_t \) and also measurable. Heuristically this means that the predictable process is known at time \( t \).

We augment the trading strategy with the price process to construct dollar valued wealth definition A.2.1.4 and gains definition A.2.1.5 stochastic processes. We restrict our analysis to the set of predictable self-financing trading strategies that admit no withdrawal or injection of funds.

**Definition A.2.1.7** (Discounted Price/Wealth/Gains Processes). We define the discounted price process \( Z^k_t = (Z^k_1, \ldots, Z^k_k) \), followed by the discounted wealth process \( V^k_t = Z^k_t \cdot \phi_t \) and discounted gains process \( G^k_t = \int_0^t \phi_u dZ^k_u \).

One of the \( k \) assets is chosen as a reference asset, deflater or numéraire such that we construct new price \( Z^k \), wealth \( W^k \) and gains \( G^k \) processes definition A.2.1.7 by dividing through by the price of the reference asset. The new processes are therefore priced relative to the price of the numéraire asset.

With this we have defined the basics of our theoretical economy under which we may price our contingent claims. The next section introduces martingales into the theory.

### A.2.2 Existence of the no-arbitrage condition and martingales

**Definition A.2.2.1** (Martingale). The expectation of a martingale \( M \) stochastic process conditioned on a filtration \( \mathcal{F}_s \) is equal to the value of the process at time \( s \). The formula expression of this is

\[
\left( \int_S |f|^p \, du \right)^{1/p}
\]

is less than infinity.
\( \mathbb{E}(M(t) | \mathcal{F}_s) = M(s) \) where \( s \leq t \). \( M \) is assumed to be (i) integrable and (ii) adapted to \( \mathcal{F}_t \).

**Definition A.2.2.2 (Local Martingale).** A measure \( \mathbb{P}^Q \) is equivalent to another measure \( \mathbb{P} \) if for all \( \omega \in \Omega \), \( \mathbb{P}^Q(\omega) = 0 \) when \( \mathbb{P}(\omega) = 0 \) and \( \mathbb{P}^Q(\omega) > 0 \) when \( \mathbb{P}(\omega) > 0 \).

**Definition A.2.2.3 (Local Martingale).** A local martingale process is a more restricted process which is only guaranteed to be a martingale up to a random stopping time \( T \).

**Definition A.2.2.4 (Arbitrage Opportunity).** An arbitrage opportunity exists when \( \mathbb{V}_0 = 0 \) but \( f(\mathbb{V}_T > 0) > 0 \) for some \( t \subset [0,T] \).

**Theorem A.2.2.1 (Fundamental Theorem of Asset Pricing One).**

We introduce the concept of a martingale definition A.2.2.1 which we use to analyse the discounted price \( \mathbb{Z}^k \), wealth \( \mathbb{V}^k \) and gains processes \( \mathbb{G}_t^k \). We examine the existence of the local martingale property definition A.2.2.3 of these processes under \( \mathbb{P}^Q \) and define \( \mathbb{P}^Q \) as an equivalent definition A.2.2.2 martingale measure if this property exists. By theorem A.2.2.1 the market is arbitrage free if and only if there exists a martingale measure for the deflated asset price under a fixed numéraire. Therefore if we assume that there is no-arbitrage in an economy with two assets, namely a bank account \( B_t \) and a stock \( S_t \), then by theorem A.2.2.1 the following holds true

\[
\frac{S_s}{B_s} = \mathbb{E}^Q \left( \frac{S_T}{B_T} | \mathcal{F}_s \right)
\]

where (i) \( \mathbb{P}^Q \) is an equivalent martingale measure (to the real world measure \( \mathbb{P} \)) and (ii) \( s \leq t \).

### A.2.3 Completeness of Markets

**Definition A.2.3.1 (Complete).** A market is complete if every contingent claim can be replicated.

**Definition A.2.3.2 (No Free Lunch with Vanishing Risk (NFLVR)).** NFLVR is a weaker condition than the no-arbitrage condition defined in definition A.2.2.4 which if it exists states that one cannot make positive profit with a loss no larger than \( \frac{1}{n} \) where \( n \) denotes the \( n \)th set of simple discrete hedging strategies. The idea is the discrete hedging strategies indexed by \( n \) will converge to a continuous time hedging strategy as \( n \to \infty \).

**Theorem A.2.3.1 (Fundamental Theorem of Asset Pricing Two).** Assume a market is arbitrage free, then the market is complete if and only if the martingale measure is unique.

### A.3 Pricing Contingent Claims with No-Arbitrage

We reformulate the problem of pricing contingent claims so that they are priced consistently in our market. We introduce an \( \mathcal{F}_t \) adapted random variable \( X_t \) to represent our contingent claim. We are interested in the set of replicating trading strategies \( \phi_t \subset \mathbb{M}^k(\mathbb{Z}_t^k, \phi_t) \) such that \( X_T = \mathbb{V}_T \).
Theorem A.3.0.2 (Martingale Representation Theorem). If $Z$ is a local martingale adapted to the filtration $\mathcal{F}_t$ generated by the Brownian motion $W$, then there exists a process $\sigma$ such that $\mathrm{d}Z = \sigma(t,\omega) \, \mathrm{d}W(t)$.

If the $\mathcal{F}_t$-adapted wealth process $V^k_t$ replicates a contingent claim $X$ and $V^k_t$ is a local martingale then the powerful result in theorem A.3.0.2 states that $X$ can be written as a stochastic integral of Brownian motion. This leaves us finally with the key result that is proven in Musiela et al. that states the wealth process of the mRNM admissible trading strategy $\phi$ that replicates $X$ is the arbitrage price $\pi(t,X)$ of $X$. What follows by Delbaen and Schachermayer Corollary 1.2 [DS94] is that

$$\pi(t,X) = Z^k(t) * \mathbf{E}^{\mathcal{P}} \left( \frac{X}{Z^k(T)} \mid \mathcal{F}(t) \right) \tag{A.3.1}$$

where (i) $Z^k$ is the numéraire asset price, (ii) the expectation is taken under the risk neutral measure and (iii) $t \leq T$. 

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Appendix B

Local Volatility and Jump Models

This section briefly reviews Local Volatility and Jump Models.

B.1 Local Volatility Models

Local volatility models first proposed by Dupire [Dup94] and Derman [DK94] parameterise the volatility function with the current value of the stock price. In the case of LIBOR market models, the forward LIBOR rate would follow the general form of

$$dL_t^T = L_t^T \sigma(t,L_t^T) W$$

where the volatility function has the additional $L_t^T$ term.

Local volatility models can further be classified into (i) CEV models and (ii) displaced diffusion models.

B.1.1 CEV

CEV models due to Cox [Cox96], postulate the form of the volatility function to be

$$\sigma(t,L_t^T) = L_t^T \beta \sigma$$

with beta restricted by $0 \leq \beta \leq 1$. The CEV models incorporate both the normal and log-normal distributions of the stock price with $\beta = 0$ and $\beta = 1$ respectively. Campbell’s [Cam87] empirical study found that the stock price and return volatility were negatively correlated providing some justification for the use of CEV properties. Andersen et al. proposed a CEV LIBOR market model extension in [AA00].

B.1.2 Displaced Diffusion

Displaced diffusion models first proposed by Rubenstein [Rub83], postulate that a process $X = L_t^T + \alpha$ was log-normally distributed. Applying this to LIBOR rates, this implies the following form of the volatility function

$$\sigma(t,L_t^T) = \frac{L_t^T + \alpha}{L_t^T} \sigma(t).$$

Rubinstein’s analysis was based on the (i) asset composition of a firm holding $a$ in risky assets and $1 - a$ in risk-less assets (leverage affect ) and (ii) the debt-equity ratio $b$. This resulted in the stock price being decomposed into risk-less and risky return components. Rebonato and Joshi combine a a displaced diffusion and stochastic LIBOR market model extension in [JR03].
Several jump extension have been proposed based on Merton’s formulation \[\text{Mer76}\] in discrete markets where he proposed a non-continuous model with jumps at discrete points. He introduced the Poisson process with the memoryless property \(^1\) such that the augmented SDE became

\[
dS = \ldots + d \left( \sum_{i=1}^{N_p(t)} (J(i) - 1) \right).
\]

Jump models are applied to LIBOR models in Glasserman and Kou \[\text{GK03}\], Glasserman and Merener \[\text{GM03}\] and Kou \[\text{Kou02}\].

\(^1\)The conditional probability given the time of the last jump is equal to the unconditional probability.
Appendix C

Change of measure

C.0.1 Introduction

A traditional analysis of a stochastic problem in the physical world might start with the modelling of behaviour in terms of a stochastic diffusion process (SDE). The parameters might be estimated or known and the model will proceed to be simulated or solved analytically under the real world measure. In the financial world and using the risk neutral approach, we change the analysis to proceed under the risk neutral measure. We define the market price of risk \( \theta(t) \), and using the Girsanov theorem we (i) introduce a new Brownian motion process and (ii) a new probability measure via the Radon-Nikodym derivative theorem and call this the risk neutral measure.

C.0.2 Justification

During the analysis of simple contingent claims under the risk neutral measure and using the non-stochastic bank account as numéraire, the calculation of the expectation proceeds fairly smoothly and the numéraire only contributes a \( dt \) term. When presented with a more complex stochastic numéraire, the expectation becomes more complicated as there is a multivariate process introduced by the interaction of the two stochastic variables. Thus knowledge of the joint distribution of the process is required to calculate the expectation. Using a change of numéraire we can remove the stochastic numéraire and replace it with a forward numéraire that is statically determined at time \( T \). The measure associated with this numéraire is called the forward measure and was introduced by Geman [Gem89].

C.0.3 Numéraire, Measures and Market Models

During our discussion of the market models we will come across four measures each with their own numéraire, namely (i) real world measure (no numéraire), (ii) the risk neutral measure (bank account), (iii) the spot measure (rolling zero bond) and (iv) the \( T \) forward measure (fixed zero bond of length \( T \)).

C.0.4 Change of Measure

Theorem C.0.4.1 (Radon-Nikodym Theorem). We define two measures \( \mathbb{P}_1 \) and \( \mathbb{P}_2 \) on the measure space \((\Omega, \mathcal{F})\). There exists a random variable \( R \) with \( \mathbb{E}^{\mathbb{P}_1} (R) = 1 \) such that \( \mathbb{P}_2 (A) = \mathbb{E}^{\mathbb{P}_1} (R 1_A) \) where \( 1_A \) is the indicator function for set \( A \).

\[ \text{Here we are talking about the simplest case of a single Brownian motion term for the numerator.} \]
Theorem C.0.4.2 (Girsanov Theorem [Gir60]). Let there be a process \( \frac{dR(t)}{R(t)} = \theta(t) dW(t) \) which is solved by the Doleans’ exponential in lemma E.0.6.3 \( \xi(R) \). If \( \theta(t) \) satisfies the Novikov condition, then the Doleans’ exponential is a martingale. Finally for all \( t \in [0,T] \), we have

\[
W^\theta(t) = W(t) + \int_0^t \theta(s) \, ds
\]

is a Brownian motion under \( \mathbb{P}_\theta \).

No-arbitrage theory has laid the groundwork for pricing under a risk neutral measure. However when postulating the SDE under the real world measure, we need additional theory that we can apply to transform both the (i) SDE and (ii) probability measure. The Radon-Nikodym Theorem theorem C.0.4.1 from measure theory provides the conditions under which an equivalent martingale will exist [ADD00, p. 65]. The Girsanov Theorem (theorem C.0.4.2) describes how the dynamics of our original SDE changes under the new measure.

Theorem C.0.4.3 (Change of numéraire). If we have two numéraires \( Z_k \) and \( Z_j \) with equivalent martingale measures \( \mathbb{P}_k \) and \( \mathbb{P}_j \), then the density of the Radon-Nikodym derivative relating the two measures is given by the maximum likelihood function

\[
\varsigma(t) = E_{t}^{\mathbb{P}_j} \left( \frac{d\mathbb{P}_k}{d\mathbb{P}_j} \right) = \frac{Z_k(t) \div Z_k(0)}{Z_j(t) \div Z_j(0)}
\]

Application of a change of measure  The following example illustrates the application of change of measure. If we have that \( \frac{X(T)}{Z_k(T)} \) is a martingale under the equivalent martingale measure \( \mathbb{Q}^{Z_k} \), then

\[
E^{Q^{Z_k}} \left( \frac{X(T)}{Z_k(T)} \right) Z_k(t) = E^{Q^{Z_j}} \left( \frac{X(T)}{Z_j(T)} \right) Z_j(t).
\]
Appendix D

Brownian Motion

D.0.5 Independent Brownian Motion

Random variables in financial modelling are traditionally modelled via basic stochastic processes such as Brownian Motion or Poisson Jump Processes. The literature is quite extensive in its coverage of both and we refer the reader to [Shr, p. 83] and [Nef00, p. 173] for more details. Ito calculus as applied to Brownian motion from a measure theory perspective is covered well in [ADD00, p. 426].

In this section we briefly define the independent Brownian motion definition D.0.5.1 as used in our SDE throughout the next two chapters. An alternative Brownian representation which is used in calibration section 6.4 is presented later.

Definition D.0.5.1 (Brownian Motion). We define the independent Brownian motion process $W^{P,Q}(t)$ (also called Wiener process) as a column vector stochastic process of size $d$ such that

$$W^{P,Q}(t) = \left( W^{P,Q}(t)_1, \ldots, W^{P,Q}(t)_d \right)$$

where (i) $dW^{P,Q}(t)_i$ is normally distributed with variance $dt$ and mean 0 under probability measure $P^Q$ and (ii) $dW_i$ and $dW_j$ are independent for $i \neq j$.

Standard usage of $W$. Let $X_i \subset [X_0, \ldots, X_N]$ be a family of stochastic processes satisfying

$$dX_i(t) = \ldots + \sigma dW$$

with $dW$ defined as in definition D.0.5.1 then (D.0.1) can be rewritten as

$$dX_i(t) = \ldots + \sum_{j=1}^{d} \sigma_{i,j}(t) dW_j(t)$$

where (i) $\sigma_{i,j}$ is the volatility function for process $X_i$ and Wiener process $W_j$ and (ii) the volatility function defined on a d-dimensional Brownian motion would be a column vector such that

$$\sigma (\ldots) \{(t,T) : 0 \leq t \leq T^* \} \times \{\omega \subset \Omega\} \rightarrow \mathbb{R}^d.$$  

D.0.6 Correlated Brownian Motion

In this section the paper presents the relationship between the uncorrelated and correlated versions of the Brownian motion process.
**Brownian Motion Transform.** Transforming a d-dimensional Brownian motion definition [D.0.5.1] into an n-dimensional correlated Brownian motion definition 6.2.1.1 is quite simple. [Bjo04] p. 59 can be consulted for more details.

We begin with the standard d-dimensional Brownian motion

\[
W = \begin{bmatrix}
W_1 \\
W_2 \\
\vdots \\
W_d
\end{bmatrix}
\]

and sigma matrix

\[
\sigma = \begin{bmatrix}
\sigma_{1,1} & \sigma_{1,2} & \cdots & \sigma_{1,d} \\
\sigma_{2,1} & \sigma_{2,2} & \cdots & \sigma_{2,d} \\
\vdots & \vdots & \ddots & \vdots \\
\sigma_{n,1} & \sigma_{n,2} & \cdots & \sigma_{n,d}
\end{bmatrix},
\]

We transform \( W \) into a n-dimensional correlated Brownian motions such that

\[
\hat{W} = \begin{bmatrix}
\hat{W}_1, W_2, \ldots, W_n
\end{bmatrix}^T
\]

\[
= \begin{bmatrix}
W_1\sigma_{1,1} + W_2\sigma_{1,2} + \cdots + W_d\sigma_{1,d} \\
W_1\sigma_{2,1} + W_2\sigma_{2,2} + \cdots + W_d\sigma_{2,d} \\
\vdots \\
W_1\sigma_{n,1} + W_2\sigma_{n,2} + \cdots + W_d\sigma_{n,d}
\end{bmatrix}
\]

\[
= \sigma W.
\]

We know the sum of *normal variates* is normally distributed with variance equal to the sum of the component normals. Defining the *Euclidean row norm* for row \( j \) as

\[
\hat{\sigma}_j = \|\sigma_j\| = \sqrt{\sum_{i=1}^{d} \sigma_{j,i}^2}
\]

we get a new matrix

\[
\hat{\sigma} = \begin{bmatrix}
\hat{\sigma}_{1,1} & \hat{\sigma}_{1,2} & \cdots & \hat{\sigma}_{1,d} \\
\hat{\sigma}_{2,1} & \hat{\sigma}_{2,2} & \cdots & \hat{\sigma}_{2,d} \\
\vdots & \vdots & \ddots & \vdots \\
\hat{\sigma}_{n,1} & \hat{\sigma}_{n,2} & \cdots & \hat{\sigma}_{n,d}
\end{bmatrix}
\]

where (i) the diagonals \( i = j \) are populated with \( \hat{\sigma}_{j,j} = \hat{\sigma}_j \) and (ii) the off diagonals \( i \neq j \) are \( \hat{\sigma}_{i,j} = \rho_{i,j} dt \) as in definition [D.0.6.1]

**Definition D.0.6.1 (Correlation Coefficient).** The correlation coefficient \( \rho_{i,j} \) between correlated Brownian motions \( \hat{W}_i \) and \( \hat{W}_j \) where \( i \neq j \) is calculated as follows:

\[
\rho_{i,j} dt = Cov[d\hat{W}_i, d\hat{W}_j] = E\left(d\hat{W}_i d\hat{W}_j\right) - E\left(d\hat{W}_i\right)E\left(d\hat{W}_j\right)
\]

\[
= E\left(d\hat{W}_i d\hat{W}_j\right) = E\left(\sum_{x=1}^{d} \sigma_{x,i} dW_x \sum_{y=1}^{d} \sigma_{j,y} dW_y\right)
\]

\[
= \sum_{x=1}^{d} \sum_{y=1}^{d} \sigma_{x,i} \sigma_{j,y} E(dW_x dW_y)
\]

\[
= \sigma_{i,1} \sigma_{j,1} + \sigma_{i,2} \sigma_{j,2} + \cdots + \sigma_{i,d} \sigma_{j,d}
\]

where we have used \( E(dW_x dW_y) = 1 \) where \( i = j \) and 0 otherwise.
More details on this can be found in [MR05, p. 415] and [Sel06].
Appendix E

Ito Calculus

The following Lemmas are useful in stochastic calculus and are presented here without proof.

**Lemma E.0.6.1** (Ito’s Product Rule). Define the two stochastic processes $X$ and $Y$ satisfying

$$dX_t = X_t \left( \alpha_t^X dt + \sigma_t^X dW \right), dY_t = Y_t \left( \alpha_t^Y dt + \sigma_t^Y dW \right)$$

where the coefficients $[\sigma, \alpha]$ satisfy [KS91, p. 289] (i) Lipschitz and (ii) linear growth conditions then there exists a strong solution satisfying

$$d(XY) = dXY + X_t dY_t + d[X, Y]. \quad (E.0.1)$$

**Lemma E.0.6.2** (Ito’s Quotient Rule). Define the two stochastic processes $X$ and $Y$ satisfying

$$dX_t = X_t \left( \alpha_t^X dt + \sigma_t^X dW \right), dY_t = Y_t \left( \alpha_t^Y dt + \sigma_t^Y dW \right)$$

where the coefficients $[\sigma, \alpha]$ satisfy [KS91, p. 289] (i) Lipschitz and (ii) linear growth conditions then there exists a strong solution satisfying

$$d \frac{X}{Y} = \frac{X}{Y} \left( \left( \alpha_t^X - \alpha_t^Y \right) dt + \left( \sigma_t^X - \sigma_t^Y \right) \left( -\sigma_t^Y dW \right) \right). \quad (E.0.2)$$

**Lemma E.0.6.3** (Dolean’s-Dade Exponential). The stochastic exponential [Pro04, p. 85] of a semi-martingale $X$ with initial condition $X_0 = 0$ written as

$$\xi(X)_t = \exp \left( X_t - \frac{1}{2} [X, X]_t \right)$$

satisfies $dZ = Z dX$ with initial condition $Z = 1$.

**Lemma E.0.6.4** (Leibniz Integral Rule). If we assume that $f(x,t)$ and $\frac{\partial f(t,x)}{\partial x}$ are continuous functions on an interval and

$$g(x) = \int_a^b f(x,t) dt,$$

then as in [Pro98, p. 179] we have

$$\frac{dg}{dx} = \int_a^b \frac{\partial f(t,x)}{\partial x} dt + \ldots.$$
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